Optimizing the Performance of D-BLAST Lattice Codes for MIMO Fading Channels

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ABSTRACT

We focus on the space-time block codes proposed in [1] for the quasi-static multi-input multi-output (MIMO) Rayleigh fading channel, referred to as the D-BLAST lattice codes. We propose a soft decision feedback decoder based on the list implementation of the sphere decoder to mitigate error propogation and demonstrate the performance improvements accrued by this decoder through simulation results. A power optimization result is established which can be employed on any full modulation diversity lattice to obtain further performance gains. We examine the coding gain achieved by the lattice codes and show that in general, constructing a full modulation diversity lattice which also optimizes the coding gain is a challenging problem.

I. INTRODUCTION

It was shown in [2] that the D-BLAST architecture could realize a significant portion of the MIMO outage capacity by employing single input single output (SISO) component codes. Coding for the D-BLAST architecture has been considered in [3, 4] and more recently in [1]. In [1] we proposed the D-BLAST lattice codes which employ SISO component codes based on algebraic number theory on this architecture. The decoding strategy involved zero forcing (ZF) or MMSE filtering to obtain the soft statistics followed by SISO decoding using the sphere decoder and had a roughly $O(K^3)$ per symbol interval average complexity at moderate to high SNR for a system with K component codes. For a system with N receive and K transmit antennas, employing K SISO constituent full modulation diversity lattice codes, it was shown through an error probability analysis (conducted for the ZF filtering case) that a diversity order of $NK - \frac{K(K-1)}{2}$ is achieved for the frame error probabilty (FEP). The analysis also yielded the coding gain (obtained earlier as the design criteria in [4]) when all component codes employed the same full modulation diversity lattice. However, the problem of obtaining full modulation diversity lattice optimizing the coding gain was not addressed. In Section VI we examine this problem and show that a large family of full modulation diversity lattices is not suitable with respect to the coding gain criteria. Nevertheless also in Section VI, we conduct a power optimization which can be used to increase the coding gain achieved by any full modulation diversity lattice.

A major hinderance in achieving performance improvements with the D-BLAST architecture is error propagation. [3] addresses this problem by employing a single trellis code which is decoded using ZF or MMSE decision feedback detection coupled with Viterbi decoding through the use of per-survivor processing. In order to keep the decoding complexity low, we consider soft decision feedback detection. Note that for the D-BLAST lattice codes, error propagation can also be reduced by increasing w, the width of the diagonal with a corresponding increase in the dimension of the lattice codes. The drawbacks of this strategy are that the decoding complexity scales as w^2 and the rate loss due to the initial set up and termination for each frame also increases. Soft decision feedback detection was also advocated in [5] but no explicit code design along with the decoding strategy to obtain the a posteriori probabilities, was suggeted. In Section V using the structure of our codes, we consider a simple, low complexity technique to obtain the soft decisions based on the list implementation of the sphere decoder [6]. Section II presents the channel model whereas Sections III and IV review the code description and the decoding respectively.

II. Channel Model

The discrete-time block fading model of a wireless communication system in a flat fading environment with N receive, K transmit antennas and a coherence interval of T symbol periods is given by

$$\mathbf{Y} = \mathbf{H}\mathbf{A}^{1/2}\mathbf{X} + \mathbf{V}\,. \tag{1}$$

$\mathbf{Y}$ is the $N \times T$ received matrix and $\mathbf{X}$ is the $K \times T$ space-time block code (STBC). The fading is described by the $N \times K$ matrix $\mathbf{H}$ having independent, identically distributed (i.i.d.), zero-mean, unit variance complex normal ($\mathcal{CN}(0,1)$) elements. The random matrix $\mathbf{H}$ stays constant for T symbol periods after which it jumps to an independent value. The $N \times T$ matrix $\mathbf{V}$ represents the additive noise at the receiver and has i.i.d., $\mathcal{CN}(0,\sigma^2)$ elements. The diagonal matrix $\mathbf{A}$ is such that the average power transmitted through transmitter k is proportional to a_k the k^{th} diagonal element of $\mathbf{A}$. Then, the code matrix $\mathbf{X}$ satisfies the power constraint $E[\mathrm{tr}(\mathbf{A}^{1/2}\mathbf{X}\mathbf{X}^{\dagger}\mathbf{A}^{1/2\,\dagger})] = T$. Thus, with our normalizations, the average received SNR (per receive antenna per symbol interval) is given by $\rho = \frac{1}{\sigma^2}$.

III. Code Description

The space time block code $\mathbf{X}$ is generated by L component encoders $\gamma_1, \cdots, \gamma_L$ operating independently. Specifically, the kK length information symbol vector (with either QAM or PAM symbols as its components) $\mathbf{u} \in \mathcal{Y}^{kK}$, with $\mathcal{Y} \subset Z[i]$, where $Z[i]$ represents the ring of complex integers, is partitioned as $\mathbf{u} = [\mathbf{u}_1^T, \cdots, \mathbf{u}_L^T]^T$ where $\mathbf{u}_m \in \mathcal{Y}^{k_m K}$ so that $k = \sum_{m=1}^{L} k_m$. $\mathbf{u}_m = [\mathbf{u}_{m,1}^T, \cdots, \mathbf{u}_{m,k_m}^T]^T$ $1 \leq m \leq L$ is then fed to the m^{th} component encoder $\gamma_m : \mathcal{Y}^{k_m K} \rightarrow \mathcal{S}^{k_m K}$ to obtain $\gamma_m(\mathbf{u}_m) = [(\mathbf{M}_m\mathbf{u}_{m,1})^T, \cdots, (\mathbf{M}_m\mathbf{u}_{m,k_m})^T]^T$. $\mathcal{S}$ represents the output symbol constellation and $\mathbf{M}_m$ is the $K \times K$ generator matrix obtained through the canonical embeddings of an algebraic number field [7–9] such that $\mathbf{M}_m$ offers full modulation diversity over $Z[i]^K$ i.e. for any $\mathbf{v}, \mathbf{w} \in Z[i]^K$ such that $\mathbf{v} \neq \mathbf{w}$ and letting

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$\mathbf{z} = \mathbf{M}_m\mathbf{v}$ and $\mathbf{y} = \mathbf{M}_m\mathbf{w}$ we have $\prod_{j=1}^{K} |z_i - y_i| > 0$. Further with some abuse of notation, we let $\gamma_m(\mathbf{u}_{m,j}) = \mathbf{M}_m\mathbf{u}_{m,j}$ for $1 \leq j \leq k_m$ and $1 \leq m \leq L$. In this work no outer code is used over the entire layer so that $\{\gamma_m(\mathbf{u}_{m,j})\}$ $1 \leq j \leq k_m$ are essentially independent. Then the output of the m^{th} encoder $\gamma_m(\mathbf{u}_m)$ is then fed to the m^{th} spatial formatter $f_m : \mathcal{S}^{k_m K} \to \tilde{\mathcal{S}}^{K \times T}$ where $\tilde{\mathcal{S}} = \{\mathcal{S}, 0\}$. f_m places the elements of $\gamma_m(\mathbf{u}_m)$ in a $K \times T$ matrix according to $\mathcal{L}_m$ the index set of the m^{th} layer (i.e. the index set for the m^{th} component code) and sets the the off-layer elements to zero. Finally defining $\mathbf{X}_m \overset{\triangle}{=} f_m(\gamma_m(\mathbf{u}_m))$ the STBC $\mathbf{X}$ is obtained as

$$\mathbf{X} = \mathbf{X}_1 + \cdots + \mathbf{X}_L \tag{2}$$

The main space-time formatting (layering) considered in this paper corresponds to the D-BLAST architecture of [2] with the width of the diagonal set to one. In particular we choose the number of layers as $L = \min(N, K)$ and we assume that $K = rL$ for some positive integer r. Then for the D-BLAST layering defining $q = r - 1$, we have that

$$k_m = \left\lfloor \frac{T - (m-1)(q+1)}{K} \right\rfloor \qquad 1 \leq m \leq L \tag{3}$$

where $\lfloor . \rfloor$ is the standard floor operator. The index set of m^{th} layer, $\mathcal{L}_m$ [1] is given by

$$\mathcal{L}_m = \{(K - g((t - (m-1)(q+1))\ mod\ K) + 1, t) : \\ 1 \leq t - (m-1)(q+1) \leq k_m K\} \tag{4}$$

where $g(.)$ is defined as

$$g(n) = \begin{cases} K & if \quad n = 0 \\ n & \qquad else \end{cases} \tag{5}$$

It can be noted that the rate of the code in bits per channel use is given by $R = \frac{kKlog(|\mathcal{Y}|)}{T}$. For typical values of the frame length in a quasi static channel (a few hundred symbol intervals) the rate loss due to the initial set up and termination can be ignored and we have that $\frac{kK}{T} \approx L$. Thus for $L = K$ the code entails almost no loss in spectral efficiency and transmits K symbols per channel use i.e. at nearly 'full rate' in the terminology of [10]. Further note that even the codes of [10, 11] transmit at most $\min(N, K)$ symbols per channel use in order to avoid exponential complexity in the decoding.

IV. Decoding

The decoding of the D-BLAST lattice codes by using the ZF as well as the MMSE filter has been discussed in [1]. Since the diversity order result and the coding gain expression are derived for the ZF filter case, we briefly describe the decoding for this case. For notational convenience, we focus on the scenario $N \geq K$ and $L = K$. We first obtain the QR decomposition of the fading matrix $\mathbf{H}$ as $\mathbf{H} = \mathbf{UQ}$ where $\mathbf{Q}$ is a $K \times K$ lower triangular matrix and the $N \times K$ matrix $\mathbf{U}$ satisfies $\mathbf{U}^\dagger\mathbf{U} = \mathbf{I}$. Without loss of generality, we assume that the matrix $\mathbf{Q}$ has positive diagonal elements. The received vector at time t, $\mathbf{Y}_t$ (the t^{th} column of the received matrix $\mathbf{Y}$ in (1)) is pre-multiplied by $\mathbf{U}^\dagger$, to obtain $\mathbf{Z}_t = \mathbf{U}^\dagger\mathbf{Y}_t$. Then we let $\mathbf{x}_m^j = \gamma_m(\mathbf{u}_{m,j})$ $1 \leq m \leq K, 1 \leq j \leq k_m$ with $x_{m,l}^j$ $1 \leq l \leq K$ denoting the l^{th} component of $\mathbf{x}_m^j$ and define $\mathbf{W} \overset{\triangle}{=} \mathbf{U}^\dagger\mathbf{V}$. Further we let $Z_{k,l}$, $Q_{k,l}$ and $W_{k,l}$ denote the $(k,l)^{th}$ elements of $\mathbf{Z}$, $\mathbf{Q}$ and $\mathbf{W}$ respectively. The vector of soft statistics for decoding $\mathbf{x}_1^1 = \gamma_1(\mathbf{u}_{1,1})$ is then given by $\tilde{\mathbf{z}}_1^1 \overset{\triangle}{=} [Z_{K,1}, \cdots, Z_{K-k+1,k}, \cdots, Z_{1,K}]^T$. Then defining $\tilde{\mathbf{Q}} = \text{diag}\{Q_{K,K}, \cdots, Q_{K-l,K-l}, \cdots, Q_{1,1}\}$, and

$$\begin{aligned} \tilde{\mathbf{A}} &= \text{diag}\{a_K, \cdots, a_{K-l}, \cdots, a_1\} \\ \tilde{\mathbf{w}}^1 &= [W_{K,1}, \cdots, W_{K-k+1,k}, \cdots, W_{1,K}]^T \end{aligned}$$

we can expand $\tilde{\mathbf{z}}_1^1$ as

$$\tilde{\mathbf{z}}_1^1 = \tilde{\mathbf{Q}}\tilde{\mathbf{A}}^{1/2}\mathbf{M}_1\mathbf{u}_{1,1} + \tilde{\mathbf{w}}^1 \overset{\triangle}{=} \mathbf{G}_{ZF}^1\mathbf{u}_{1,1} + \tilde{\mathbf{w}}^1 \tag{6}$$

Note that the noise vector $\tilde{\mathbf{w}}^1$ has i.i.d $\mathcal{CN}(0, \sigma^2)$ elements. The sphere decoder [12] is used on the soft statistics (6) with $\mathbf{G}_{ZF}^1$ as the equivalent generator matrix, to obtain $\hat{\mathbf{u}}_{1,1}$, the decision for $\mathbf{u}_{1,1}$. As noted in [9], the expected complexity of the sphere decoder (at moderate to high SNR) is $O(K^3)$. Further as in the case of [9], due to the form of (6), using the sphere decoder with a real generator matrix $\mathbf{M}_1$ is about 4 times less complex than that with a complex matrix $\mathbf{M}_1$ but the resulting performance is generally poorer. With $\hat{\mathbf{u}}_{1,1}$, we obtain $\mathbf{M}_1\hat{\mathbf{u}}_{1,1}$ which is fed back. The decoding of $\gamma_2(\mathbf{u}_{2,1}), \cdots, \gamma_K(\mathbf{u}_{K,1}), \gamma_1(\mathbf{u}_{1,2}), \cdots$ is done assuming perfect feedback, in a similar manner and the process continues till the entire frame is decoded. [2] It can be verified that the expected implementation complexity (per symbol interval) at moderate to high SNR is of the order $O(K^3)$ which compares favourably to the $O(K^5)$ complexity for the codes of [10, 11].

V. Soft Decision Feedback decoding

In order to mitigate error propagation effects, in this section we consider a soft decision feedback decoder. Again for notational convenience, we assume that $N \geq K$ and that $L = K$. Consider the decoding of $\mathbf{x}_m^j = \gamma_m(\mathbf{u}_{m,j})$. We let $\mathbf{C} \overset{\triangle}{=} \mathbf{HA}^{1/2}$ and let $\mathbf{X}_l$ denote the l^{th} column of $\mathbf{X}$ with $X_{k,l}$ denoting the $(k,l)^{th}$ element of $\mathbf{X}$. Further, we assume that $\mathbf{x}_m^j$ occurs in the block code $\mathbf{X}$ in the interval t to $t + K - 1$ i.e. we have that

$$X_{K-k+1,t+k-1} = x_{m,k}^j \quad 1 \leq k \leq K\ . \tag{7}$$

We define $\mu_{k,l} \overset{\triangle}{=} E[X_{k,l}]$ and $\beta_{k,l} \overset{\triangle}{=} E[|X_{k,l} - \mu_{k,l}|^2]$ to be the mean and variance of $X_{k,l}$ respectively. The a-priori mean values are equal to zero and the a-priori value of the variance $\beta_{k,l}$ is denoted by $\alpha_{k,l}$. The K soft statistics for decoding $\mathbf{x}_m^j$ are then obtained as

$$z_{m,k}^j = \mathbf{F}_k^\dagger \left(\mathbf{Y}_{t+k-1} - \sum_{l=1}^{K-k} \mathbf{C}_l \mu_{l,t+k-1} \right) \quad 1 \leq k \leq K \tag{8}$$

where $\mathbf{Y}_l$ and $\mathbf{C}_l$ denotes the l^{th} column of $\mathbf{Y}$ and $\mathbf{C}$ respectively . The filter $\mathbf{F}_k$ is given by

$$\mathbf{F}_k = \left(\sigma^2\mathbf{I} + \mathbf{CB}_{(k,t)}\mathbf{C}^\dagger\right)^{-1}\mathbf{C}_{K-k+1} \tag{9}$$

with the matrix $\mathbf{B}_{(k,t)}$ being,

$$\mathbf{B}_{(k,t)} = \text{diag}\{\beta_{1,t+k-1}, \cdots, \beta_{K,t+k-1}\} \tag{10}$$

Note that in the matrix $\mathbf{B}_{(k,t)}$ the variances $\beta_{j,t+k-1}$, $K-k+1 \leq j \leq K$ are set to their a-priori values $\alpha_{j,t+k-1}$. The soft statistics are then modified to obtain,

$$\tilde{z}_{m,k}^j = \frac{z_{m,k}^j \exp(-i\phi_k)}{\sqrt{\sigma^2\mathbf{F}_k^\dagger\mathbf{F}_k + \sum_{l \neq K-k+1} |\mathbf{F}_k^\dagger\mathbf{C}_l|^2 \beta_{l,t+k-1}}} \tag{11}$$

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[1] $(a,t) \in \mathcal{L}_m$ means that the symbol transmitted at the t^{th} interval through transmitter a belongs to layer m

[2] Decoding the diagonal sections corresponding to the frame termination require some straightforward modifications

where, $\mathbf{F}_k^\dagger \mathbf{C}_{K-k+1} = |\mathbf{F}_k^\dagger \mathbf{C}_{K-k+1}| \exp(i\phi_k)$. Then, letting $\delta_{m,k}^j \overset{\triangle}{=} \frac{|\mathbf{F}_k^\dagger \mathbf{C}_{K-k+1}|}{\sqrt{\sigma^2 \mathbf{F}_k^\dagger \mathbf{F}_k + \sum_{l \neq K-k+1} |\mathbf{F}_k^\dagger \mathbf{C}_l|^2 \beta_{l,t+k-1}}}$, $1 \leq k \leq K$, we can simplify (11) as

$$\tilde{z}_{m,k}^j = \delta_{m,k}^j x_{m,k}^j + \tilde{\eta}_{m,k}^j \ . \tag{12}$$

Further, we let $\tilde{\mathbf{z}}_m^j = [\tilde{z}_{m,1}^j, \cdots, \tilde{z}_{m,K}^j]^T$, $\tilde{\boldsymbol{\eta}}_m^j = [\tilde{\eta}_{m,1}^j, \cdots, \tilde{\eta}_{m,K}^j]^T$ and $\mathbf{G}_m^j = \text{diag}\{\delta_{m,1}^j, \cdots, \delta_{m,K}^j\}\mathbf{M}_m$ to obtain the vector model,

$$\tilde{\mathbf{z}}_m^j = \mathbf{G}_m^j \mathbf{u}_{m,j} + \tilde{\boldsymbol{\eta}}_m^j \tag{13}$$

We assume the vector $\tilde{\boldsymbol{\eta}}_m^j$ to have i.i.d $\mathcal{CN}(0,1)$ elements. The vector $\tilde{\mathbf{z}}_m^j$ is fed to the sphere decoder which uses $\mathbf{G}_m^j$ as the effective generator matrix to obtain a list of N_c closest points to the vector $\tilde{\mathbf{z}}_m^j$. We let $\hat{\mathbf{v}}_s, \ 1 \leq s \leq N_c$ denote the information symbol vectors obtained so that $\{\mathbf{G}_m^j \hat{\mathbf{v}}_s\}, \ 1 \leq s \leq N_c$ are the N_c closest points obtained. Note that this list implementation requires a simple modification [6] and as claimed in [6] results in only a small increase in complexity. Further, the squared distances $||\tilde{\mathbf{z}}_m^j - \mathbf{G}_m^j \hat{\mathbf{v}}_s||^2$ are obtained as a by product of the algorithm. The decision $\hat{\mathbf{u}}_{m,j}$ is given as

$$\hat{\mathbf{u}}_{m,j} = \arg \min_{\hat{\mathbf{v}}_s : 1 \leq s \leq N_c} ||\tilde{\mathbf{z}}_m^j - \mathbf{G}_m^j \hat{\mathbf{v}}_s||^2 \tag{14}$$

Then, defining $\hat{\mathbf{w}}_s = \mathbf{M}_m \hat{\mathbf{v}}_s, \ 1 \leq s \leq N_c$ we obtain the a-posteriori probabilities [3] as,

$$\Pr(\mathbf{x}_m^j = \hat{\mathbf{w}}_s | \tilde{\mathbf{z}}_m^j) = \frac{\exp(-||\tilde{\mathbf{z}}_m^j - \mathbf{G}_m^j \hat{\mathbf{v}}_s||^2)}{\sum_{l=1}^{N_c} \exp(-||\tilde{\mathbf{z}}_m^j - \mathbf{G}_m^j \hat{\mathbf{v}}_l||^2)} \quad 1 \leq s \leq N_c \tag{15}$$

Further, letting $\hat{w}_{k,s}$ deonte the k^{th} $1 \leq k \leq K$ component of $\hat{\mathbf{w}}_s$ we note that since full modulation diversity lattices are used for each layer we have that the event $x_{m,k}^j = \hat{w}_{k,s}$ is identical to the event $\mathbf{x}_m^j = \hat{\mathbf{w}}_s$ and hence $\Pr(x_{m,k}^j = \hat{w}_{k,s} | \tilde{\mathbf{z}}_m^j)$ for $1 \leq s \leq N_c$ are also given by (15). Based on the a-posteriori probabilities $\Pr(x_{m,k}^j = \hat{w}_{k,s} | \tilde{\mathbf{z}}_m^j)$ where $1 \leq k \leq K$ and $1 \leq s \leq N_c$ we obtain the new vales of the means and variances $\mu_{K-k+1,t+k-1}$ and $\beta_{K-k+1,t+k-1}$ for $1 \leq k \leq K$, which are then used to decode the subsequent diagonals.

VI. Error Probability Analysis

An error probability analysis was conducted for the D-BLAST lattice codes decoded using the ZF filter, in [1]. We briefly present the main results which will then be used in sub-sections VI-A and VI-B. For convenience, we consider a K transmit antenna system with K layers and $N \geq K$. We expand the average FEP obtained using the ZF filter and denoted by $\Pr(\mathcal{E})$ as

$$\Pr(\mathcal{E}) = \Pr(\mathcal{E}_1 \cup \mathcal{E}_2 \cdots \cup \mathcal{E}_K) \ , \tag{16}$$

with $\Pr(\mathcal{E}_m)$ being the average error probabolity of layer m. Further using the arguments developed in [1], we have

$$\Pr(\mathcal{E}) = \Pr(\mathcal{E}_1^g \cup \mathcal{E}_2^g \cdots \cup \mathcal{E}_K^g) \ , \tag{17}$$

where $\Pr(\mathcal{E}_m^g)$ denotes the average error probability of layer m (component code m) under perfect feedback. Then the pair wise error probability (PWEP) between $\mathbf{x}_m = \gamma_m(\mathbf{u}_m)$ and $\mathbf{s}_m = \gamma_m(\mathbf{v}_m)$ corresponding to the m^{th}, $1 \leq m \leq K$ layer in a genie aided system can be upper bounded as

$$\Pr(\mathbf{x}_m \to \mathbf{s}_m) \leq \frac{1}{(1 + \frac{a_K \sum_{j=1}^{k_m} |x_{m,1}^j - s_{m,1}^j|^2}{4\sigma^2})^N \cdots (1 + \frac{a_1 \sum_{j=1}^{k_m} |x_{m,K}^j - s_{m,K}^j|^2}{4\sigma^2})^{N-K+1}}$$

Note that using a full modulation diversity lattice for layer m guarantees that for every pair $(\mathbf{x}_m, \mathbf{s}_m)$, $\sum_{j=1}^{k_m} |x_{m,l}^j - s_{m,l}^j|^2 > 0 \quad 1 \leq l \leq K$, hence the diversity order achieved for $\Pr(\mathcal{E}_m^g)$ doted by $\mathcal{D}_m$ is given by

$$\mathcal{D}_m = N + N - 1 + \cdots + N - K + 1 = NK - \frac{K(K-1)}{2} \tag{18}$$

Thus if each layer uses a full modulation diversity lattice, we have from (17) that since $\Pr(\mathcal{E}_m^g) \leq \Pr(\mathcal{E}) \leq \sum_{k=1}^{K} \Pr(\mathcal{E}_k^g)$, the diversity order achieved for the FEP denoted by $\mathcal{D}$ is also given by (18). Further from [1], we define the coding gain for layer m as

$$\min_{\substack{\mathbf{y}_m = \mathbf{M}_m \mathbf{t}_m, \mathbf{z}_m = \mathbf{M}_m \mathbf{r}_m \\ \mathbf{t}_m \neq \mathbf{r}_m, \mathbf{t}_m, \mathbf{r}_m \in \mathcal{Y}^K}} \left((1 + a_K \theta_1)^N \cdots (1 + a_1 \theta_K)^{N-K+1}\right)^{1/\mathcal{D}_m} \tag{19}$$

where $\theta_l, \ 1 \leq l \leq K$ is given as

$$\theta_l = \frac{|y_{m,l} - z_{m,l}|^2}{4\sigma^2} \tag{20}$$

Note that due to the symmetry resulting from the D-BLAST layering considered here, the pairwise error analysis conducted above is valid for all layers and hence the design criteria for choosing the optimum generator matrix according to (19) remains the same across all layers and we thus assume $\mathbf{M} = \mathbf{M}_m, \ 1 \leq m \leq K$. Further, in this paper we focus on the high SNR regime and obtain the coding gain for the system as

$$\min_{\substack{\mathbf{y} = \mathbf{M}\mathbf{t}, \mathbf{z} = \mathbf{M}\mathbf{r} \\ \mathbf{t} \neq \mathbf{r}; \mathbf{t}, \mathbf{r} \in \mathcal{Y}^K}} (\quad a_K^N \cdots a_1^{N-K+1})^{1/\mathcal{D}} \times \frac{|y_1 - z_1|^{2N/\mathcal{D}} \cdots |y_K - z_K|^{2(N-K+1)/\mathcal{D}}}{4\sigma^2} \tag{21}$$

A. Power Optimization

We now consider the problem of maximizing the coding gain (21) by optimally selecting the coefficients $a_k, \ 1 \leq k \leq K$. Since the power constraint considered is $E[\text{tr}(\mathbf{A}^{1/2}\mathbf{X}\mathbf{X}^\dagger \mathbf{A}^{1/2 \ \dagger})] = T$, we assume that T along with the values $w_k \overset{\triangle}{=} E[\mathbf{X}^k \mathbf{X}^{k \ \dagger}], \ 1 \leq k \leq K$, where $\mathbf{X}^k$ is the k^{th} row of $\mathbf{X}$, have been provided. Then, we consider the maximization problem

$$\max_{\substack{a_1, \cdots, a_K : a_k > 0 \ 1 \leq k \leq K; \\ \sum_{k=1}^K w_k a_k = T}} \left(a_1^{N-K+1} \cdots a_K^N\right)^{1/\mathcal{D}} \tag{22}$$

We first transform the problem in (22) to a convex optimization problem. Note that since $0 < \frac{N-k+1}{\mathcal{D}} < 1, \ 1 \leq k \leq K$, $a_m^{\frac{N-k+1}{\mathcal{D}}}$ is a concave funtion of a_m for $a_m > 0$. Further the geometric mean $(x_1 \cdots x_K)^{1/K}$ is a concave function of $x_1, \cdots, x_K$ for $x_k > 0, \ 1 \leq k \leq K$. Then using the results on concavity of composite functions [13], it can be verified that the function $(a_K^N \cdots a_1^{N-K+1})^{\frac{1}{K\mathcal{D}}}$ is a concave function of its arguments when each $a_k > 0$. Hence we consider the equivalent concave maximization problem

$$\max_{\substack{a_1, \cdots, a_K : a_k > 0 \ 1 \leq k \leq K; \\ \sum_{k=1}^K w_k a_k = T}} \left(a_1^{N-K+1} \cdots a_K^N\right)^{\frac{1}{K\mathcal{D}}} \tag{23}$$

Now since any solution to the Kuhn-Tucker (KT) conditions [13] for (23) is also an optimum solution, employing the KT conditions we obtain the optimum values of a_k where $1 \leq k \leq K$ denoted by a_k^* as

$$a_k^* = \frac{(N-K+k)T}{w_k \mathcal{D}} \quad 1 \leq k \leq K \tag{24}$$

Further letting $||\mathbf{m}_k||^2$ denote the squared norm of the k^{th} row of $\mathbf{M}$, since $w_k = c||\mathbf{m}_{K-k+1}||^2$ where c is a constant, using (24) in (21) we obtain the new coding gain expresion as

$$\min_{\substack{\mathbf{y} = \mathbf{M}\mathbf{t}, \mathbf{z} = \mathbf{M}\mathbf{r} \\ \mathbf{t} \neq \mathbf{r}; \mathbf{t}, \mathbf{r} \in \mathcal{Y}^K}} \tilde{c} \left(\frac{|y_1 - z_1|}{||\mathbf{m}_1||}\right)^{2N/\mathcal{D}} \cdots \left(\frac{|y_K - z_K|}{||\mathbf{m}_K||}\right)^{2(N-K+1)/\mathcal{D}} \tag{25}$$

where $\tilde{c}$ is a constant involving the given parameters N, K, T, σ^2 and the average energy of constellation $\mathcal{Y}$

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[3] Note that the following simple rule is used due the absence of any outer code

B. Optimum Lattice

We now consider the problem of determining the optimum full modulation diversity lattice, which maximizes the coding gain (25). We seek to construct a full modulation diversity lattice that is good (i.e. yields a large coding gain) for a range of input constellation sizes $|\mathcal{Y}|$ where $\mathcal{Y} \subset Z[i]$. Note that such an approach may not yield a lattice optimum with respect to a particular information constellation $\mathcal{Y}$, but an exhaustive search for generator matrices maximizing (25) rapidly gets intractable for even moderate values of $|\mathcal{Y}|$ and K. Hence we first consider the term

$$\mathcal{X}_{\mathbf{M}} = \inf_{\substack{\mathbf{v} \in \Lambda \\ \mathbf{v} \neq \mathbf{O}}} |v_1|^{2N/\mathcal{D}} \cdots |v_K|^{2(N-K+1)/\mathcal{D}} \tag{26}$$

where Λ is the lattice generated as $\mathbf{M}Z[i]^K$. We seek to obtain a family of matrices $\{\mathbf{M}\}$ with $\mathcal{X}_{\mathbf{M}} > 0$. Note that $\mathcal{X}_{\mathbf{M}}$ may be thought of as the unnormalized coding gain obtained with $\mathbf{M}$. A similar approach for finding good lattices for the perfectly interleaved Rayleigh fading, single transmit antenna channel was adopted in [7, 14] and more recently in [15]. The unnormalized coding gain expression for that channel involves the minimum product distance defined as

$$\min_{\substack{\mathbf{v} \in \Lambda \\ \mathbf{v} \neq \mathbf{O}}} |v_1|^2 \cdots |v_K|^2 \tag{27}$$

For both the cases $\Lambda = \mathbf{M}Z^K$ and $\Lambda = \mathbf{M}Z[i]^K$ [7, 14, 15] provide full modulation diversity lattices obtained through canonical embeddings of algebraic number fields, which ensure (27) has a value equal to 1. After ensuring a non-zero value of (27), [14, 15] maximize the coding gain by minimizing the average energy. Unfortunately for our channel for general N and K, as proved in the theorem below, for a large family of full modulation diversity lattices, (26) is equal to 0 and finding full modulation diversity lattices which ensure a non-zero value of (26) with $\Lambda = \mathbf{M}Z[i]^K$ is a hard problem. We let F be an algebraic number field of dimension K over Q (Q denotes the field of rational numbers) and consider the case $\Lambda = \mathbf{M}Z[i]^K$. Let O_F denote the ring of algebraic integers of F and $\omega_1, \cdots, \omega_K$ be an integral basis so that $O_F = Z\omega_1 + \cdots + Z\omega_K$ and $\sigma_1(.), \cdots, \sigma_K(.)$ be the K canonical embeddings of F to C. We assume that the signature of F is (s, t) i.e. $\sigma_1(.), \cdots, \sigma_s(.)$ correspond to the real embeddings and the rest to complex conjugate pairs[4] and let $\mathcal{N}(\alpha)$ denote the norm of $\alpha \in F$. Further, in order to ensure full modulation diversity we assume $s \geq 1$ so that $Q(i)$ is not a sub-field of F [15]. We let $\pi(.)$ be a permutation operator on $\{1, \cdots, K\}$ so that the generator matrix $\mathbf{M}$ is constructed as $M_{jk} = \sigma_{\pi(j)}(\omega_k)$. Then we have,

$$\begin{aligned} \mathcal{X}_{\mathbf{M}} \quad \leq \quad \mathcal{X}'_{\mathbf{M}} &= \inf_{\substack{\alpha \in O_F \\ \alpha \neq \mathbf{O}}} |\sigma_{\pi(1)}(\alpha)|^{2N/\mathcal{D}} \cdots |\sigma_{\pi(K)}(\alpha)|^{2(N-K+1)/\mathcal{D}} \\ &= \inf_{\substack{\alpha \in O_F \\ \alpha \neq \mathbf{O}}} \left(|\sigma_1(\alpha)|^{\beta_1} \cdots |\sigma_{s+t}(\alpha)|^{\beta_{s+t}} \right)^{2/\mathcal{D}} \end{aligned} \tag{28}$$

where since $N \geq K$ we have $\beta_k \geq 1,\ 1 \leq k \leq s$ and $\beta_k \geq 2,\ s+1 \leq k \leq s+t$.

Theorem 1: $\mathcal{X}'_{\mathbf{M}} = 1$ if and only if $\beta_k = \beta$ for all $1 \leq k \leq s$ and $\beta_k = 2\beta$ for all $s+1 \leq k \leq s+t$, where β is some positive integer. For all other cases $\mathcal{X}'_{\mathbf{M}} = 0$.

Proof: Note that if the condition is satisfied then $\mathcal{X}'_M = \inf_{\substack{\alpha \in O_F \\ \alpha \neq \mathbf{O}}} |\mathcal{N}(\alpha)|^{2\beta/\mathcal{D}} = 1$. Now suppose the condition is violated. Let $2\beta_m = \min\{2\beta_1, \cdots, 2\beta_s, \beta_{s+1}, \cdots, \beta_{s+t}\}$ for some $1 \leq m \leq s$.[5] Then letting $\{\epsilon\}$ denote the group of units of O_F we have

$$\mathcal{X}'_{\mathbf{M}} \leq \inf_{\alpha \in \{\epsilon\}} \left(\prod_{\substack{j=1 \\ j \neq m}}^{s} |\sigma_j(\alpha)|^{\beta_j - \beta_m} \prod_{k=s+1}^{s+t} |\sigma_k(\alpha)|^{\beta_k - 2\beta_m} \right)^{2/\mathcal{D}} \tag{29}$$

Now let $l(\alpha) = [\ln|\sigma_1(\alpha)|, \cdots, \ln|\sigma_s(\alpha)|, \cdots, \ln|\sigma_{s+t}(\alpha)|^2]$ denote the logarithimic representation of $\alpha \in \{\epsilon\}$. Then using the property that the group $\{l(\alpha) : \ \alpha \in \{\epsilon\}\}$ forms a *full* lattice in the subspace $\mathcal{L} \overset{\triangle}{=} \{[\lambda_1, \cdots, \lambda_{s+t}] \in \mathcal{R}^{s+t} : \sum_{k=1}^{s+t} \lambda_k = 0\}$ [16], we can readily show that the right hand side of (29) and hence $\mathcal{X}'_{\mathbf{M}}$ is equal to 0. ■

Note that as an immediate consequence of the theorem we have $\mathcal{X}_{\mathbf{M}} = \mathcal{X}'_{\mathbf{M}} = 0$ whenever $s > 1$, since then the condition required in the theorem is always violated. In general even if $\mathcal{X}'_{\mathbf{M}} > 0$, ensuring that the lattices also yield $\mathcal{X}_{\mathbf{M}} > 0$ is still an open problem. Further for the simple case with $s = 1$ and $\Lambda = \mathbf{M}Z^K$ we can readily find a permutation to satisy the condition of the theorem and since here $\mathcal{X}_{\mathbf{M}} = \mathcal{X}'_{\mathbf{M}}$ we have $\mathcal{X}_{\mathbf{M}} = 1$. In this paper we restrict our attention to the full modulation diversity lattices provided in [7, 15]. Through simulation examples we show the performance improvements obtained through power optimization done on those lattices. Hence the error probabilities provided here serve as an upper bound to the best attainable error rates.

VII. Simulation Results

FEP comparisons between the D-BLAST lattice codes and the codes of [10, 11] have been carried out in [1] for various values of N and K at different spectral efficiencies. It was shown that the performance achieved by the D-BLAST lattice codes was generally close to that achieved by the codes of [10, 11] but at a lower decoding complexity. Here we demonstrate the further improvements that can achieved with the D-BLAST lattice codes through soft decision feedback and power optimization. In Fig. 1 we plot the average symbol error probability (SEP) of a D-BLAST lattice code for a three transmit and receive antenna ($N = K = 3$) example with three layers. The 3×3 complex cyclotomic rotation [7] is chosen as the full modulation diversity generator matrix for all the layers and we set $\mathbf{A} = c\mathbf{I}$ where c is the normalizing constant. The information symbol constellation $\mathcal{Y} \subset Z[j]$ where $j \overset{\triangle}{=} \exp(i2\pi/3)$, has size 4 and the frame length considered is $T = 102$ symbol intervals so that the total rate is (almost) 6 bits per channel use (PCU)[6]. The SEPs achieved on using the ZF as well as the MMSE filter are plotted. Also plotted is the SEP achieved by the soft decision feedback decoder of Section V with $N_c = 16$. Note that at the SEP of 10^{-3} the soft decision feedback decoder gains about 1 and 3 dB compared to its MMSE and ZF couterpart respectively. To demonstrate the benefits of power optimiztion presented in Section VI, we consider a system with $N = K = 2$ and two layers in Fig 2. The generator matrix used is the 2×2 complex cyclotomic rotation [7]. The symbol constellation $\mathcal{Y} \subset Z[i]$ is taken to be the rectangular 4 QAM constellation with $T = 200$ symbol intervals so that

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[4] Note that $K = s + 2t$

[5] The case $\beta_m = \min\{2\beta_1, \cdots, 2\beta_s, \beta_{s+1}, \cdots, \beta_{s+t}\}$, $s+1 \leq m \leq s+t$ can be handled similarly

[6] The actual rate is about 5.88 bits PCU due to the set up and termination loss.

the total rate is (almost) 4 bits PCU. We plot the FEP achieved by using the ZF filter, for three choices of the matrix $\mathbf{A}$ given by, $\mathbf{A}_1 = c \times \mathrm{diag}\{4/5, 1/5\}$,$\mathbf{A}_1 = c \times \mathrm{diag}\{3/4, 1/4\}$ and $\mathbf{A}^* = c \times \mathrm{diag}\{1/3, 2/3\}$ where c is the normalizing constant and $\mathbf{A}^*$ denotes the optimal choice. Note that the diversity order of all the FEPs is 3 and at the FEP of 10^{-2} the optimized design gains about 1.3 and 2 dB compared to the design with $\mathbf{A}_2$ and $\mathbf{A}_1$ respectively.

VIII. CONCLUSION

We considered the D-BLAST lattice codes and proposed a simple low complexity soft decision feedback decoder based on the list implementation of the sphere decoder, to mitigate error propogation. We established a power optimization result which can be employed on any full modulation diversity lattice to obtain performance gains and examined the problem of obtaining optimum full modulation diversity lattices for the D-BLAST lattice codes. Performance gains measured in terms of SEP and FEP were demonstrated through simulation results.

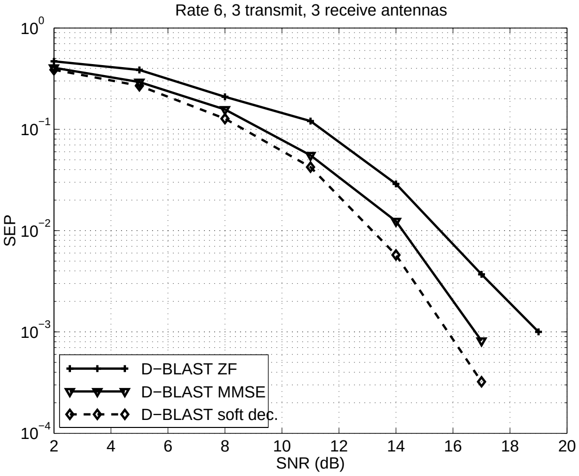

Fig. 1.

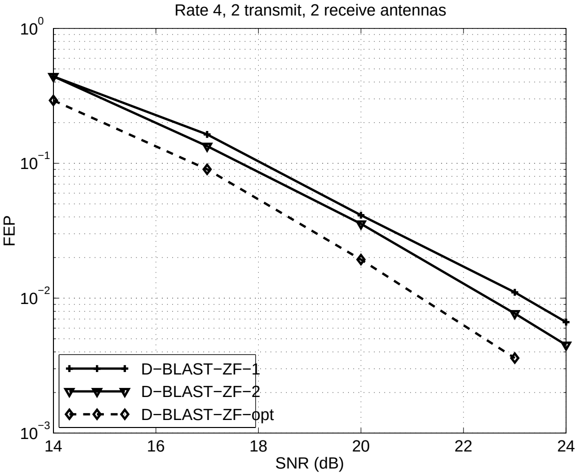

Fig. 2.

REFERENCES

[1] N. Prasad and M. K. Varanasi, "D-blast lattice codes for mimo block rayleigh fading channels," in *Proc. Allerton Conf. on Comm., Control, and Comput.*, Monticello, IL, Oct. 2002, University of Illinois.

[2] G. J. Foschini, "Layered space–time architecture for wireless communication in fading environments when using multiple antennas," *Bell Labs Tech. J.*, vol. 1, no. 2, pp. 41–59, Autumn 1996.

[3] G. Caire and G. Colavolpe, "On space-time coding for quasistatic multiple-antenna channels," submitted to *IEEE Trans. Inform. Theory*, 2001.

[4] D. Shiu and J. M. Kahn, "Layered space-time codes for wireless communications using multiple transmit antennas," in *Proc. IEEE Intl. Conf. on Communications*, Vancouver, Canada, June 1999.

[5] D. Shiu and J. M. Kahn, "Scalable layered space-time codes for wireless communications: Performance analysis and design criteria," in *Proc. Wireless Communications and Networking Conf.*, New Orleans, LA, Sept. 1999, invited.

[6] B. Hochwald and S. ten Brink, "Achieving near-capacity on multiple-antenna channel," submitted to *IEEE Trans. Commun.*, July 2001, also in *Proc. Allerton Conf. on Comm. Control, and Comput.*, Oct. 2001.

[7] X. Giraud, E. Boutillon, and J.-C. Belfiore, "Algebraic tools to build modulation schemes for fading channels," *IEEE Trans. Inform. Theory*, vol. 43, no. 3, pp. 938–952, May 1997.

[8] J. Boutros and E. Viterbo, "Signal space diversity: A power- and bandwidth-efficient diversity technique for the rayleigh fading channel," *IEEE Trans. Inform. Theory*, vol. 44, no. 4, pp. 1453–1467, July 1998.

[9] M. O. Damen, K. Abed-Meraim, and J.-C. Belfiore, "Diagonal algebraic space–time block codes," *IEEE Trans. Inform. Theory*, vol. 4, no. 3, pp. 628–636, Mar. 2002.

[10] H. El Gamal and M. O. Damen, "Universal space–time coding," submitted to *IEEE Trans. Inform. Theory*, Jan. 2002.

[11] B. Hassibi and B. M. Hochwald, "High-rate codes that are linear in space and time," *IEEE Trans. Inform. Theory*, vol. 48, no. 7, pp. 1804–1824, July 2002.

[12] M. O. Damen, A. Chkeif, and J.-C. Belfiore, "Lattice code decoder for space–time codes," *IEEE Commun. Letters*, vol. 4, no. 5, pp. 161–163, May 2000.

[13] S. Boyd and L. Vandenberghe, *Convex Optimization*, Course notes for Convex Optimization with Engineering Applications, www.stanford.edu/class/ee364, 2001.

[14] J. Boutros, E. Viterbo, C. Rastello, and J.-C. Belfiore, "Good lattice constellations for both rayleigh fading and gaussian channels," *IEEE Trans. Inform. Theory*, vol. 42, no. 2, pp. 502–518, Mar. 1996.

[15] P. Dayal and M. K. Varanasi, "An improved algebraic family of complex lattices for single transmit antenna Rayleigh fading with application to Universal Space–Time Codes," in *Proc. Allerton Conf. on Comm., Control, and Comput.*, Monticello, IL, Oct. 2002.

[16] Z. I. Borevich and I. R. Shafarevich, *Number Theory*, Academic Press, Orlando, Florida, 1966.