Throughput analysis for MIMO systems in the high SNR regime

Narayan Prasad and Mahesh K. Varanasi
Electrical & Computer Engineering Department
University of Colorado, Boulder, CO 80309

Abstract — **Outage capacity and throughput are the two key metrics through which the fundamental limits of delay-sensitive wireless MIMO links can be studied. In this paper, we show that these metrics are intimately related, and consequently, as in the case of outage capacity, the growth rate of throughput with SNR ρ is $t \log \rho$ for a general class of fading channels (with channel state information at the receiver (CSIR) and with or without CSI at the transmitter (CSIT)) whose channel matrix is of rank t with probability one. However, while asymptotically tight affine lower bounds of the form $t \log \rho + O(1)$ were recently derived for outage capacity for such channels, in the sense that the limit as $\rho \to \infty$ of the difference between the outage capacity and the lower bound is zero, such affine lower bounds are not possible in general for the throughput. Using the $t \log \rho + O(1)$ bounds on outage capacity however, lower bounds on throughput are specified where the high SNR limit of the ratio of the throughput and its lower bound is unity. These bounds reveal that the throughput optimal outage probability approaches zero as $\rho \to \infty$. An important exception is the scenario where both the transmitter and receiver have CSI under the long-term power constraint (LTPC), for which we obtain a lower bound of the form $t \log \rho + O(1)$ which is asymptotically tight (in the stronger sense) and interestingly, this lower bound is identical to the asymptotic delay-limited capacity. The throughputs of MISO and SIMO fading channels are extensively analyzed and it is shown that asymptotically, isotropic Gaussian input is throughput optimal, correlation is detrimental whereas increase in the Rice factor is beneficial and that throughput is schur-concave in the correlation eigenvalues.**

I. Introduction

Consider delay-constrained MIMO systems, where each transmitted codeword experiences only one fading realization. The channel state information (CSI) is known perfectly to the receiver (perfect CSI-R) but may or may not be known to the transmitter. For such systems, *outage capacity* is a relevant figure of merit [1, 2] and is defined as the maximum rate that can be transmitted reliably over all channel realizations, except a subset whose probability is less than a (acceptable) pre-specified fraction. The outage capacity formulation leaves open the question of what constitutes an optimal operating point or value for the probability of error. This question is addressed by considering *throughput* which is another key performance metric in delay-limited packetized communications. It represents the average rate which can be reliably transmitted to the receiver allowing for the possibility that some data may be incorrectly decoded. In particular, we consider a notion of throughput that applies to delay-sensitive systems as well as to reliability sensitive systems under certain constraints on complexity of the receiver. In each case, we consider the two scenarios in which the transmitter may or may not have CSI (i.e. with or without CSIT). We show in this paper that the outage capacity and the throughput at any SNR are in general (for problems that involve no CSIT and those that do including the short-term as well as the long-term power constrained versions) related in that the throughput is the supremum (over outage probability) of the product of the fraction of correctly decoded packets and the outage capacity. This allows us to apply the many results available in the analysis of outage capacity in the literature, see for instance [3] and the references therein, to characterize the behavior of throughput.

A large class of fading channels was considered by the authors in [3] that encompasses any fading distribution satisfying the twin conditions of the channel matrix $\mathbf{H}$ being finite and of rank t, $1 \leq t \leq \min(N, K)$, with probability one. For this class of channels, *asymptotically tight* (in the limit of high SNR) affine lower bounds on the outage capacities (of the form $t \log(\rho) + O(1)$) were obtained. By asymptotically tight bound we mean that the high SNR limit of the difference between the actual capacity and its lower bound is identically equal to zero. The constant term ($O(1)$) provides a much finer characterization of the outage capacity. For instance, it captures the effect of the channel distribution and is sensitive to the presence of CSI at the transmitter. When applied to our result on throughput, the $t \log(\rho) + O(1)$ type asymptotically tight affine lower bounds yield a lower bound (denoted as $\tilde{\mathcal{T}}_\rho$) on throughput that is tight in the sense that the limit as $\rho \to \infty$ the ratio of the throughput to its lower bound $\tilde{\mathcal{T}}_\rho$ is equal to unity. These bound reveals that the throughput scales as $t \log(\rho)$ and that the (throughput) optimal outage probability goes to zero as SNR increases. Moreover, for MISO and SIMO systems the bound is also tight in the stronger sense.

In general however, the throughput does not behave as $t \log(\rho) + O(1)$ at high SNR as does outage capacity. There is an important exception, however. This is the case of CSIT with a long-term power constraint over the so-called "regular" fading channels (channels for which the expected value of the reciprocal of the geometric mean of the channel eigenvalues is finite). In this case, we obtain a key result in this paper which is that the delay-limited capacity [3, 4] is an asymptotically tight lower bound on throughput for regular channels with CSIT-LTPC. Moreover, the behavior of the delay-limited capacity for high SNR is like [3, 4] $t \log(\rho) + t \log \left(\frac{1}{t E[\frac{1}{\lambda^{gm}}]} \right)$ where λ^{gm} is the geometric mean of the eigenvalues of the

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channel matrix. Finally, to illustrate the throughput computations, we analyze the throughput of the correlated Rayleigh and Rician fading channels with particular emphasis on the MISO and SIMO cases. The correlated Rayleigh fading channel is modeled as $\mathbf{H} = \mathbf{A}^{1/2}\mathbf{H}_w\mathbf{B}^{1/2}$, where $\mathbf{A} \geq 0$ satisfies $\mathrm{tr}(\mathbf{A}) = N$ and $\mathbf{B} \geq 0$ satisfies $\mathrm{tr}(\mathbf{B}) = K$. The elements of the $N \times K$ matrix, $\mathbf{H}_w$ are i.i.d. $\mathcal{CN}(0,1)$. The Rician channel with rank-1 mean matrix, henceforth referred to as just the Rician channel, is modeled as $\mathbf{H} = \sqrt{1-r}\mathbf{H}_w + \sqrt{r}\hat{\mathbf{H}}$, for some $r \in [0,1)$ and where the rank-1 deterministic matrix $\hat{\mathbf{H}}$ satisfies $||\hat{\mathbf{H}}||_F^2 = NK$.

II. System Model

We consider a MIMO block-fading channel with N receive and K transmit antennas, governed by a discrete-time baseband model

$$\mathbf{y}_j = \mathbf{H}\mathbf{x}_j + \mathbf{v}_j,\ j = 1, 2, \cdots, J. \tag{1}$$

$\mathbf{x}_j$ is the $K \times 1$ channel input at time j which satisfies $E[||\mathbf{x}_j||^2] \leq \rho$ and $\mathbf{y}_j$ is the corresponding $N \times 1$ channel output. The noise vector $\mathbf{v}_j$ is spatially and temporally white and is taken to have i.i.d. ($\mathcal{CN}(0,1)$) elements. The $N \times K$ random channel matrix $\mathbf{H}$ stays constant for J symbol intervals after which it jumps to an independent value. It is assumed that $\mathbf{H}$ is perfectly known to the receiver. We consider the scenario when the transmitter has perfect CSI as well as the case where the transmitter has no CSI but nevertheless knows the distribution of $\mathbf{H}$. In the latter case, unless otherwise stated, we make a uniform power assumption i.e. we set $E[\mathbf{x}\mathbf{x}^\dagger] = \frac{\rho}{K}\mathbf{I}$.

The throughput results obtained here are valid for any fading distribution where $\mathbf{H}$ is of rank t, $1 \leq t \leq \min(N, K)$, with probability one and where the throughput [1] is finite for all finite SNRs.[2] These twin conditions evidently include a very general class of fading channels. For instance, they include the usual correlated Rayleigh and Rician fading channel models. Moreover, the virtual channel representation of realistic scattering environments [5] often results in a fading matrix (denoted by $\mathbf{H}_V$ in [5]) that satisfies our conditions for some $t < \min(N, K)$ with probability one.

III. Outage Capacity: A Review

In this section, we summarize briefly two key results on outage capacity obtained in [3] along with two new ones, that allow us to make a self-contained presentation as well as to establish notation and render it convenient to discuss the relationships between outage capacity and throughput. The case of CSIT with STPC as well as CSIR without the uniform power assumption are omitted for brevity. The analysis for the ergodic capacity in the CSIR-only case has been presented in [6].

A. CSI at Receiver

With no CSIT and under the uniform power restriction the outage probability $\Pr(\mathcal{O})$ at rate R and a given power ρ, is given by $\Pr(\mathcal{O}) = \Pr\left(\log\left(\left|\mathbf{I} + \frac{\rho}{K}\mathbf{H}\mathbf{H}^\dagger\right|\right) < R\right)$. For any $\epsilon \in (0,1)$, the resulting outage capacity, denoted $\mathcal{C}_\rho^{ML}(\epsilon)$, is

$$\mathcal{C}_\rho^{ML}(\epsilon) = \sup\left\{R : \Pr\left(\log\left(\left|\mathbf{I} + \frac{\rho}{K}\mathbf{H}\mathbf{H}^\dagger\right|\right) < R\right) \leq \epsilon\right\}.$$

Next, let $\pi_k = \{a_1, \cdots, a_k\}$ be any subset of $\{1, \cdots, N\}$ of size k and let the i^{th} row of $\mathbf{H}$ be denoted as $\mathbf{g}_i$ and define $\mathbf{G}_{\pi_k}$ as a matrix obtained from $\mathbf{H}$ by striking out rows with indices not in π_k, i.e. let

$$\mathbf{H} = \begin{bmatrix} \mathbf{g}_1 \\ \vdots \\ \mathbf{g}_N \end{bmatrix}, \quad \mathbf{G}_{\pi_k} = \begin{bmatrix} \mathbf{g}_{a_1} \\ \vdots \\ \mathbf{g}_{a_k} \end{bmatrix},\ 1 \leq k \leq N. \tag{2}$$

We have the following theorem.

Theorem 1: A lower bound to $\mathcal{C}_\rho^{ML}(\epsilon)$ is given by

$$\mathcal{C}_\rho^{ML}(\epsilon) \geq \tilde{\mathcal{C}}_\rho^{ML}(\epsilon) \triangleq t\log(\rho) + \log(\hat{\mathcal{C}}_\infty^{ML}(\epsilon)), \tag{3}$$

where

$$\hat{\mathcal{C}}_\infty^{ML}(\epsilon) = \sup\left\{y : \Pr\left(\sum_{\pi_t} |\frac{1}{K}\mathbf{G}_{\pi_t}\mathbf{G}_{\pi_t}^\dagger| < y\right) \leq \epsilon\right\}. \tag{4}$$

$\tilde{\mathcal{C}}_\rho^{ML}(\epsilon)$ is asymptotically tight in that $\lim_{\rho\to\infty}(\mathcal{C}_\rho^{ML}(\epsilon) - \tilde{\mathcal{C}}_\rho^{ML}(\epsilon)) = 0$.

The uniform power assumption is asymptotically optimal when $t = K$.

B. CSI at Transmitter and Receiver with LTPC

We now consider the outage capacity with perfect CSIT and a long term power constraint (LTPC) [4, 7]. Let $\lambda_1 \geq \lambda_2 \geq \cdots, \lambda_t > 0$ be the t ordered positive eigenvalues of $\mathbf{H}^\dagger\mathbf{H}$ having a *continuous joint density function*. Further, let $\lambda^{gm} \triangleq (\prod_{k=1}^t \lambda_k)^{1/t}$ denote their geometric mean. Then $\mathcal{C}_\rho^{ML-LTPC}(\epsilon)$ is is equal to [4, 7]

$$\sup\left\{R : \min_{E[\sum_{k=1}^t \gamma_k]\leq\rho}\left\{\Pr\left(\log\prod_{k=1}^t(1+\lambda_k\gamma_k) < R\right)\right\} \leq \epsilon\right\}.$$

Setting $\mathcal{C}_\rho^{ML-LTPC}(\epsilon) = t\log(\rho) + \log(\hat{\mathcal{C}}_\rho^{ML-LTPC}(\epsilon))$ we get

Theorem 2: An asymptotically tight lower bound to $\mathcal{C}_\rho^{ML-LTPC}(\epsilon)$ is given by

$$\tilde{\mathcal{C}}_\rho^{ML-LTPC}(\epsilon) \triangleq t\log(\rho) + \log(\hat{\mathcal{C}}_\infty^{ML-LTPC}(\epsilon)),$$

where

$$\hat{\mathcal{C}}_\infty^{ML-LTPC}(\epsilon) = \lim_{\rho\to\infty}\hat{\mathcal{C}}_\rho^{ML-LTPC}(\epsilon)$$

$$= \sup\left\{y : \min_{E[\tilde{\gamma}]\leq\frac{1}{t}}\left\{\Pr\left(\tilde{\gamma}\lambda^{gm} < y^{1/t}\right)\right\} \leq \epsilon\right\}. \tag{5}$$

The constant $\hat{\mathcal{C}}_\infty^{ML-LTPC}(\epsilon)$ is determined by the system of equations

$$\Pr\left(\lambda^{gm} < \frac{t\ (\hat{\mathcal{C}}_\infty^{ML-LTPC}(\epsilon))^{1/t}}{\hat{z}}\right) = \epsilon,$$

$$E\left[\frac{t\ (\hat{\mathcal{C}}_\infty^{ML-LTPC}(\epsilon))^{1/t}}{\lambda^{gm}}\mathcal{X}\left(\lambda^{gm} \geq \frac{t\ (\hat{\mathcal{C}}_\infty^{ML-LTPC}(\epsilon))^{1/t}}{\hat{z}}\right)\right] = 1,$$

[1] It will be precisely defined in the sequel.

[2] It is possible to construct valid distributions where the throughput is infinite. However, then the corresponding ergodic capacities would also be infinite and such cases are physically meaningless.

where $\mathcal{X}(\cdot)$ denotes the indicator function.
For correlated Rayleigh as well as the Rician channel, simple but accurate expressions for the constants in the affine lower bounds can be obtained via the Gaussian approximations.[3] For convenience we assume $t = K$ and offer the following proposition.

Proposition 1: Assuming $\ln\left|\frac{1}{K}\mathbf{H}^\dagger\mathbf{H}\right|$ to be $\mathcal{N}(\mu, \sigma^2)$, we have that

$$\log(\hat{\mathcal{C}}_\infty^{ML}(\epsilon)) \approx (\mu + \sigma\phi^{-1}(\epsilon))\log(e)$$
$$\log(\hat{\mathcal{C}}_\infty^{ML-LTPC}(\epsilon)) \approx \left(\mu - \frac{\sigma^2}{2K}\right)\log(e) - K\log\left(1 - \phi\left(\frac{\sigma}{K} + \phi^{-1}(\epsilon)\right)\right), \quad (6)$$

where $\phi(.)$ denotes the cdf of a standard normal random variable.
The mean and variance needed for the above proposition have been derived in many works, see for instance [8], which considers correlated Rician channels.

C. Delay Limited Capacities

For fading channels which also satisfy, $E[\frac{1}{\lambda^{gm}}] < \infty$, termed "regular" fading channels in [4], an asymptotically tight lower bound to the delay limited capacity,

$$\mathcal{C}_\rho^{ML-DL} \triangleq \lim_{\epsilon\to 0^+} \mathcal{C}_\rho^{ML-LTPC}(\epsilon), \quad (7)$$

was obtained in [4]. That bound, denoted here by $\tilde{\mathcal{C}}_\rho^{ML-DL}$, also follows from Theorem 2. Note that the correlated Rayleigh and Rician channels are regular when $\max(rank(\mathbf{A}), rank(\mathbf{B})) \geq 2$ and $\max(N, K) \geq 2$, respectively.

Theorem 3: An asymptotically tight lower bound to $\mathcal{C}_\rho^{ML-DL}$ is given by

$$\tilde{\mathcal{C}}_\rho^{ML-DL} = t\log(\rho) + t\log\left(\frac{1}{tE[\frac{1}{\lambda^{gm}}]}\right). \quad (8)$$

Note that the Gaussian approximation for $\log(\hat{\mathcal{C}}_\infty^{ML-LTPC}(\epsilon))$ is consistent in that in the limit $\epsilon \to 0$ it goes to $\log(\hat{\mathcal{C}}_\infty^{ML-DL}) = K\log\left(\frac{1}{KE[\frac{1}{\lambda^{gm}}]}\right)$. The following theorem characterizes the asymptotic effect of correlation.

Theorem 4: $\tilde{\mathcal{C}}_\rho^{ML-DL}$ is maximized by $\mathbf{A} = \mathbf{I}_N$ and $\mathbf{B} = \mathbf{I}_K$.

Proof: Suppose $N \geq K$. From (8) we see that without loss of optimality we can assume $rank(\mathbf{A}) \geq K$ & $\mathbf{B} > 0$, so that

$$\tilde{\mathcal{C}}_\rho^{ML-DL} = K\log(\rho) - K\log(K) - K\log(E[|\mathbf{H}^\dagger\mathbf{H}|^{-1/K}]) =$$
$$K\log(\rho) - K\log(K) + \log(|\mathbf{B}|) - K\log(E[|\mathbf{H}_w^\dagger\mathbf{A}\mathbf{H}_w|^{-1/K}]). \quad (9)$$

Then in (9), using the AM-GM inequality we get, $\mathbf{I}_K = \arg\max_{\mathbf{B}>0,\ \mathrm{tr}(\mathbf{B})\leq K}\{|\mathbf{B}|\}$. Also since $\mathbf{H}_w$ is invariant to unitary transformations, without loss of generality we can restrict $\mathbf{A}$ to be a diagonal positive semi-definite matrix with at least K positive elements. Let $\mathbf{A}^\Pi = \Pi\mathbf{A}\Pi^\dagger$ for any permutation matrix Π. Then noting that $|.|^{-1/K}$ is convex over set of positive definite matrices and that Π is unitary, we get

$$E[|\mathbf{H}_w^\dagger\mathbf{A}\mathbf{H}_w|^{-1/K}] \geq E[|\mathbf{H}_w^\dagger\tilde{\mathbf{A}}\mathbf{H}_w|^{-1/K}] \quad (10)$$

where $\tilde{\mathbf{A}} = (\sum_\Pi \mathbf{A}^\Pi)/N! = \alpha\mathbf{I}$, for some $\alpha > 0$. Since $\mathrm{tr}(\tilde{\mathbf{A}}) = \mathrm{tr}(\mathbf{A})$, we get that

$$\mathbf{I}_N = \arg\min_{\mathbf{A}>\mathbf{0},\ \mathrm{tr}(\mathbf{A})\leq N}\{E[|\mathbf{H}_w^\dagger\mathbf{A}\mathbf{H}_w|^{-1/K}]\}. \quad (11)$$

A similar argument works when $N < K$. ■

[3]These approximations are provably tight in large array regime but are very effective even for practical array sizes.

IV. THROUGHPUT

In this section, we consider throughput as the performance metric. It represents the average rate which can be reliably transmitted to the receiver allowing for the possibility that some data may be incorrectly decoded. In particular, we consider a notion of throughput that applies to delay-sensitive systems as well as to reliability sensitive systems under certain constraints on complexity of the receiver. In each case, we consider the two scenarios in which the transmitter may or may not have CSI (i.e. with or without CSIT). For delay-sensitive systems without CSIT, the incorrectly decoded packet (determined via a CRC check) is dropped. For delay-sensitive systems with CSIT, no transmission takes place during the outage state (and hence the packet is not reliably decoded) and the scheduled packet is dropped by the transmitter. On the other hand, in reliability sensitive systems without CSIT, the incorrectly decoded packet is retransmitted by the transmitter which in turn is notified by the receiver via an ACK/NACK bit. The re-transmissions take place till the packet is successfully decoded and the underlying idea is that the wireless channel cannot stay in the outage state forever. In reliability sensitive systems with CSIT no transmission takes place during the outage state and this time the transmitter waits for the next non-outage state which can support the desired rate. Note the contrast to the delay-limited capacity formulation where no outage states are allowed.

For convenience, we assume that the channel changes independently after each packet transmission and that in case of re-transmissions (of the same codeword) the receiver only uses the most recent received packet for decoding. The latter assumption models a complexity constrained receiver and yields a conservative estimate of the achievable throughput. With these assumptions the delay-sensitive throughput is identical to the reliability-sensitive throughput (the latter is referred to as the maximum zero-outage throughput in [9]).

We now proceed to conduct a high-SNR analysis of the throughput. Mathematically, throughput is defined as $\mathcal{T}_\rho = \sup\{R(1 - \Pr(\mathcal{O}))\}$, where $\Pr(\mathcal{O})$ denotes the outage probability at rate R and SNR ρ. Before proceeding, we make the natural assumption that $\mathcal{T}_\rho < \infty$ for all finite ρ. Then letting $\mathcal{C}_\rho(\epsilon)$ denote the corresponding outage capacity (generically, for CSIR only, CSIT-STPC or CSIT-LTPC problems), we have the following result.

Theorem 5: The throughput and the corresponding outage capacity satisfy the relation

$$\begin{aligned} \mathcal{T}_\rho &= \sup_{\epsilon\in(0,1)} \{(1-\epsilon)\mathcal{C}_\rho(\epsilon)\} \\ &= \sup_{\epsilon\in(0,1)} \{(1-\epsilon)(t\log(\rho)+\log(\hat{\mathcal{C}}_\rho(\epsilon)))\}. \quad (12) \end{aligned}$$

Thus a lower bound to the the throughput is given by

$$\mathcal{T}_\rho \geq \tilde{\mathcal{T}}_\rho = \sup_{\epsilon\in(0,1)} \{(1-\epsilon)(t\log(\rho)+\log(\hat{\mathcal{C}}_\infty(\epsilon)))\}. \quad (13)$$

Proof: First note that

$$\begin{aligned} \mathcal{T}_\rho &= \sup_{\epsilon\in(0,1)} \sup\{R(1-\Pr(\mathcal{O})) : \Pr(\mathcal{O}) \leq \epsilon\} \\ &\geq \sup_{\epsilon\in(0,1)} \{(1-\epsilon)\sup\{R : \Pr(\mathcal{O}) \leq \epsilon\}\} \\ &= \sup_{\epsilon\in(0,1)} \{(1-\epsilon)\mathcal{C}_\rho(\epsilon)\}\,. \quad (14) \end{aligned}$$

Now since $\mathcal{T}_\rho$ is finite, $\forall\ \alpha > 0,\ \exists\ R$ such that $R(1-\Pr(\mathcal{O})) \geq \mathcal{T}_\rho - \alpha$. For that R, set $\epsilon = \Pr(\mathcal{O})$ so that $(1-\epsilon)R \geq \mathcal{T}_\rho - \alpha$. Hence, we have that

$$\sup_{\epsilon\in(0,1)} \{(1-\epsilon)\mathcal{C}_\rho(\epsilon)\} \geq (1-\epsilon)R \geq \mathcal{T}_\rho - \alpha \quad (15)$$

Using (14), and noting that (15) holds for arbitrarily small α, the theorem is proved. ■

The expressions for $\hat{\mathcal{C}}_\infty(\epsilon)$ can be obtained from Theorem 1 for the CSIR-only problem and from Theorem 2 for the CSIT problem under LTPC to obtain lower bounds on the throughput in the two cases, respectively. In each of these cases, using the lower bound in (13), it can be deduced that

$$\lim_{\rho\to\infty} \frac{\mathcal{T}_\rho}{\log(\rho)} = t, \ \ \& \ \ \lim_{\rho\to\infty} \frac{\mathcal{T}_\rho}{\tilde{\mathcal{T}}_\rho} = 1. \quad (16)$$

Further insights can be gained on the throughput scaling using the alternate expression in (12) and is summarized in the following theorem.

Theorem 6: For the CSIR-only as well as the CSIT-STPC scenarios the throughput scales as $\mathcal{T}_\rho = t\log(\rho) + \log(\hat{\mathcal{T}}_\rho)$, where $\log(\hat{\mathcal{T}}_\rho) = o(\log(\rho))$ i.e. $\lim_{\rho\to\infty} \frac{\log(\hat{\mathcal{T}}_\rho)}{\log(\rho)} = 0$ but $\lim_{\rho\to\infty} \log(\hat{\mathcal{T}}_\rho) = -\infty$. The throughput optimal outage $\hat{\epsilon}_\rho \to 0$ as $\rho \to \infty$. For MISO and SIMO cases the lower bound in (13) is tight in the stronger sense.

Proof: The argument given below applies to both the scenarios. Let $\hat{R}_\rho$ and $\hat{\epsilon}_\rho$ denote the throughput-optimal rate and outage probability, respectively, so that $\mathcal{T}_\rho = (1-\hat{\epsilon}_\rho)\hat{R}_\rho$. In (12), since at any fixed $\epsilon \in (0,1)$, $\hat{\mathcal{C}}_\rho(\epsilon)$ decreases as $\rho \to \infty$, we can conclude that $\hat{\epsilon}_\rho \to 0$ as $\rho \to \infty$. Also, using the alternate expression in (12), we have that $\hat{R}_\rho = t\log(\rho) + \log(\hat{\mathcal{C}}_\rho(\hat{\epsilon}_\rho))$. Invoking (16), we see that $\hat{R}_\rho = t\log(\rho) + o(\log(\rho))$. Similarly, we have $\mathcal{T}_\rho = t\log(\rho) + o(\log(\rho))$, where now $o(\log(\rho)) = -\hat{\epsilon}_\rho \log(\rho) + (1-\hat{\epsilon}_\rho)\log(\hat{\mathcal{C}}_\rho(\hat{\epsilon}_\rho))$. Also, since for any fixed ϵ, $\hat{\mathcal{C}}_\rho(\epsilon)$ decreases with ρ to $\hat{\mathcal{C}}_\infty(\epsilon)$ and $\lim_{\epsilon\to 0} \hat{\mathcal{C}}_\infty(\epsilon) = 0$, for both cases $\lim_{\rho\to\infty} o(\log(\rho)) = -\infty$, so that neither the throughput nor the throughput-optimal rate permit an asymptotically tight affine lower bound of the form $t\log(\rho) + O(1)$. Finally, for the MISO and SIMO case, using the arguments given above with the fact that now $\hat{\mathcal{C}}_\rho(\epsilon) = \frac{1}{\rho} + \hat{\mathcal{C}}_\infty(\epsilon)$ we can also conclude that the lower bound in (13) is tight in the stronger sense. ■

However, a sharper statement can be made for regular channels with perfect CSIT and long-term power control which is captured in the following result.

Theorem 7: Delay-limited capacity is an asymptotically tight lower bound on the throughput for regular channels with CSIT and long term power control. Thus, for such systems we have that

$$\lim_{\rho\to\infty} (\mathcal{T}_\rho - \mathcal{C}_\rho^{ML-DL}) = \lim_{\rho\to\infty} (\tilde{\mathcal{T}}_\rho - \tilde{\mathcal{C}}_\rho^{ML-DL}) = 0 \quad (17)$$

where $\mathcal{C}_\rho^{ML-DL}$ is defined in (7) and the asymptotically tight lower bound $\tilde{\mathcal{C}}_\rho^{ML-DL}$ for it is given in Theorem 3.

Proof: For regular channels with CSIT using (12) the throughput can be written as

$$\mathcal{T}_\rho = \sup_{\epsilon\in(0,1)} \{(1-\epsilon)(t\log(\rho)+\log(\hat{\mathcal{C}}_\rho^{ML-LTPC}(\epsilon)))\} \quad (18)$$

Since $\hat{\mathcal{C}}_\rho^{ML-LTPC}(0) = \hat{\mathcal{C}}_\rho^{ML-DL} > 0$ and for all $\epsilon \in [0,1)$, $\hat{\mathcal{C}}_\rho^{ML-LTPC}(\epsilon)$ is non-increasing in ρ, we get that

$$\mathcal{T}_\rho \geq \tilde{\mathcal{C}}_\rho^{ML-DL} = t\log(\rho) + \log(\hat{\mathcal{C}}_\infty^{ML-DL}) \quad (19)$$

and that the solution in (18) $\hat{\epsilon}_\rho \to 0$ as $\rho \to \infty$. As a consequence, for any $\epsilon_o > 0$ arbitrarily small, $\exists\ \rho_o$ large enough such that $\forall\ \rho \geq \rho_o$

$$\mathcal{T}_\rho \leq t\log(\rho) + \log(\hat{\mathcal{C}}_\rho^{ML-LTPC}(\epsilon_o)). \quad (20)$$

Set $x = 1/\rho$ and note for regular channels the difference $\log(\hat{\mathcal{C}}_{1/x}^{ML-LTPC}(\epsilon)) - \log(\hat{\mathcal{C}}_\infty^{ML-DL})$ is continuous in the compact set $(x,\epsilon) \in [0,1] \times [0,\delta]$ (any $\delta < 1$) and hence it is uniformly continuous. Using this fact, we have that given any $\alpha > 0$, for x, δ small enough

$$\log(\hat{\mathcal{C}}_{1/x}^{ML-LTPC}(\epsilon)) - \log(\hat{\mathcal{C}}_\infty^{ML-DL}) \leq \alpha, \ \ \forall\ \epsilon \in [0,\delta]\,. \quad (21)$$

Combined with (20) this means that $\forall \rho > \rho_o$

$$\mathcal{T}_\rho \leq t\log(\rho) + \log(\hat{\mathcal{C}}_\infty^{ML-DL}) + \alpha\,. \quad (22)$$

The bounds in (22) and (19) prove the theorem. ■

In other words, for such systems the high-SNR throughput-optimal rate and outage probability pair is $(\mathcal{C}_\rho^{ML-DL}, 0)$. Moreover since the throughput with CSIT and STPC and that with only CSIR are lower that the throughput with CSIT and LTPC, using Theorem 7 we can conclude that the delay-limited capacity is an *asymptotic upper bound* for the throughput of regular channels without LTPC.

V. MISO and SIMO Fading Channels

Note that in the case of the CSIT problem with a long-term power constraint, the analysis of high SNR throughput reduces to the analysis of the delay-limited capacity. In [3], we showed that for the rank-one Rician fading channel, the delay-limited capacity increases with Rice-factor for SIMO or MISO channels whereas it decreases with Rice-factor when

$\min\{N, K\} \geq 2$. Theorem 7 of course implies that the asymptotic throughput for the CSIT problem with LTPC inherits this exact behavior. Similarly, the asymptotic effect of correlation on the delay limited capacity characterized by Theorem 4 also applies to the asymptotic throughput. It is not only the case of CSIT with LTPC that benefits from the throughput analysis in this paper. In this section, we demonstrate this by considering the CSIR-only problem in the context of the MISO and SIMO fading channels. Letting $\boldsymbol{\mu}$ denote the correlation eigenvalues, we prove the following theorem

Theorem 8: For MISO and SIMO correlated Rayleigh fading channels (a) there exists an $\hat{\epsilon}$ such that $\hat{\mathcal{C}}_{\infty}^{ML}(\epsilon)$ is schur-concave in $\boldsymbol{\mu}$ when $\epsilon \in (0, \hat{\epsilon}]$, (b) $\forall \epsilon \in (0, \hat{\epsilon}]$, the uncorrelated channel yields the maximum $\hat{\mathcal{C}}_{\infty}^{ML}(\epsilon)$ and (c) the uniform power allocation is optimal for the i.i.d Rayleigh fading channel.

Proof: We first consider the MISO case and characterize the effect of the transmit correlation eigenvalues on $\hat{\mathcal{C}}_{\infty}^{ML}(\epsilon)$. We define $\tilde{\boldsymbol{\mu}} = \boldsymbol{\mu}/K$ so that $\sum_{k=1}^{K} \tilde{\mu}_k = 1$. Let $\hat{\epsilon}$ be such that

$$\hat{\epsilon} \stackrel{\triangle}{=} \min_{\substack{\tilde{\mu} \in \mathbb{R}_K^+ \\ \sum_{k=1}^K \tilde{\mu}_k = 1}} \Pr\left(\sum_{k=1}^{K} \tilde{\mu}_k |h_{w,k}|^2 < 1\right). \tag{23}$$

Then, it is clear that for any given set of correlation eigenvalues $\{\mu_k\}$ $\hat{\mathcal{C}}_{\infty}^{ML}(\hat{\epsilon}) \leq 1$, so that for all $\epsilon \in (0, \hat{\epsilon}]$, $\hat{\mathcal{C}}_{\infty}^{ML}(\epsilon) \leq 1$. Consequently, we have that $\forall\ \epsilon \in (0, \hat{\epsilon}]$, $\hat{\mathcal{C}}_{\infty}^{ML}(\epsilon)$ equals

$$\sup\left\{y : y \in (0,1]\ \&\ \Pr\left(\sum_{k=1}^{K} \tilde{\mu}_k |h_{w,k}|^2 < y\right) \leq \epsilon\right\}, \tag{24}$$

Now, a result in [10, 11] says that $\Pr(\sum_{k=1}^{K} \tilde{\mu}_k |h_{w,k}|^2 < y)$ is schur-convex in $\tilde{\boldsymbol{\mu}}$ for all $y \leq 1$. Using this result we see that $\hat{\mathcal{C}}_{\infty}^{ML}(\epsilon)$ in (24) is schur-concave in $\boldsymbol{\mu}$ when $\epsilon \in (0, \hat{\epsilon}]$. It can be shown that $\forall y \in (0, 1]$ (see [10] and problem 5.16 in [12])

$$\min_{\substack{\tilde{\mu} \in \mathbb{R}_K^+ \\ \sum_{k=1}^K \tilde{\mu}_k = 1}} \Pr\left(\sum_{k=1}^{K} \tilde{\mu}_k |h_{w,k}|^2 < y\right) = \Pr\left(\sum_{k=1}^{K} |h_{w,k}|^2 < Ky\right)$$
$$= 1 - \exp(-Ky) \sum_{k=0}^{K-1} \frac{(Ky)^k}{k!} \tag{25}$$

so that $\hat{\epsilon}$ can be determined from (25) setting $y = 1$. Further, it can be readily proved that the same $\hat{\epsilon}$ works for the SIMO case as well as $\hat{\mathcal{C}}_{\infty}^{ML-STPC}(\epsilon)$. The remaining facts stated in the theorem follow as a consequence of (25) and (24). ■

For the Rician MISO and SIMO channels we offer the following result without proof.

Theorem 9: For MISO and SIMO Rician channels, the Rice factor is beneficial in the sense that there exists $\bar{\epsilon}$ such that $\frac{d}{dr}\hat{\mathcal{C}}_{\infty}^{ML}(\epsilon) > 0$ for all $\epsilon \in (0, \bar{\epsilon}]$.
As a byproduct of Theorem 6 and Theorems 8, 9 we can obtain the following result on the behavior of throughput for the MISO and SIMO fading channels.

Theorem 10: The high SNR throughputs for the MISO and SIMO correlated Rayleigh fading channels with only CSIR and uniform power allocation are *schur-concave* in the correlation eigenvalues and asymptotically, the presence of correlation is detrimental to the throughput. Further, the same results also hold for the case of CSIT with a short-term power constraint. Furthermore, for the MISO i.i.d Rayleigh fading channel uniform power allocation is asymptotically throughput optimal. Moreover, asymptotically an increase in the Rice factor is beneficial for the throughputs (CSIR-only as well as CSIT-STPC) of MIMO and SIMO channels.

VI. CONCLUSIONS

The relation between outage capacity and the throughput was captured. Based on this it was shown that the throughout optimal outage probability goes to zero as SNR increases and that the throughput can be expanded as $t\log(\rho)+o(\log(\rho))$ but in general does not allow an asymptotically tight affine bound. However, in the case of regular channels with CSIT and LTPC an asymptotically tight affine lower bound, which is also tight for the delay-limited capacity, is possible. The throughputs of the MISO (and SIMO) channels were also extensively analyzed and it was shown that asymptotically, isotropic Gaussian input is throughput optimal, correlation is detrimental whereas increase in the rice factor is beneficial and that the throughput is schur-concave in the correlation eigenvalues.

REFERENCES

[1] G. J. Foschini and M. J. Gans, "On limits of wireless communications in a fading environment when using multiple antennas," *Wireless Personal Communications*, vol. 6, no. 3, pp. 311–335, Mar. 1998.
[2] E. Biglieri, J. Proakis, and S. Shamai, "Fading channels: Information-theoretic and communication aspects," *IEEE Trans. Inform. Theory*, vol. 44, pp. 2619–2692, Oct. 1998.
[3] N. Prasad and M. K. Varanasi, "Mimo outage capacity in the high snr regime," in *Proc. IEEE Intl. Symposium on Inform. Theory*, Adelaide, Australia, 2005, pp. 656–660.
[4] E. Biglieri, G. Caire, and G. Taricco, "Limiting performance of block-fading channels with multiple antennas," *IEEE Trans. Inform. Theory*, vol. 47, no. 4, pp. 1273–1289, May 2001.
[5] A. M. Sayeed, "Deconstructing multiantenna fading channels," *IEEE Trans. Signal Processing*, vol. 50, no. 10, pp. 2563–25796, Oct. 2002.
[6] A. Lozano, A. M. Tulino, and S. Verdu, "High-snr offset in the multi-antenna communication," *IEEE Trans. Inform. Theory*, vol. 51, no. 12, pp. 4134–4151, Dec. 2005.
[7] G. Caire, G. Taricco, and E. Biglieri, "Optimum power control over fading channels," *IEEE Trans. Inform. Theory*, vol. 45, no. 5, pp. 1468–1489, July 1999.
[8] M. R. Mckay and I. B. Collings, "General capacity bounds for spatially correlated rician mimo channels," *IEEE Trans. Inform. Theory*, vol. 51, no. 9, pp. 3121–3145, Sept. 2005.
[9] N. Ahmed, M. A. Khojastepour, and R. Baraniuk, "Delay-limited throughput maximization in fading channels using rate and power control," in *Proc. IEEE Global Telecommun. Conf.*, Dec. 2004, pp. 3459–3463.
[10] H. Boche and E. A. Jorsweick, "Outage probability of multiple antenna systems: Optimal transmission and impact of correlation," in *Proc. Intl. Zurich Seminar on Communications*, Zurich, Switzerland, Feb. 2004, pp. 116–119.
[11] H. Boche and E. A. Jorswieck, "On Schur-convexity of expectation of weighted sum of random variables with applications," *Journal of inequalities in pure and applied mathematics*, vol. 5, no. 2, 2004.
[12] D. N. C. Tse and P. Viswanath, *Fundamentals of Wireless Communication*, Cambridge University Press, 2005.