The Simple Relay Channel: Optimum Resource Allocation and Capacity

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Abstract— **A simple relay channel has multiple simple relays assisting communication between a single source-destination pair. A simple relay has a half-duplex constraint that cannot, in addition, decode signals sent to it by more than one source and/or relay nodes simultaneously over the same frequency band. This paper analyzes a general 2-parameter (power and bandwidth) protocol (which we call the simple protocol) for the simple relay channel and analyzes the achievable rate using a decode-forward strategy and proposes (a) the optimization of the achievable rate over the two parameters (which is shown to be a concave optimization problem) and (b) derives conditions under which the achievable rate coincides with the max-flow min-cut upper bound, thereby characterizing its capacity under those conditions. In the special case of a single relay channel, the simple relay is just a half-duplex relay and the simple protocol is somewhat more general than the one considered in [1].**

I. Introduction

The orthogonal relay channel is one in which the relays transmit and receive in channels that are non-interfering to each other. The relays used over such channels have been called cheap or half-duplex relays. This is an important constraint to impose, since in its absence, the signal transmitted from the relay would interfere with its own received signal. In [2], the authors compared SNR losses of orthogonal relay channels having channel state information at receiver (CSIR) alone, with those that do not have the half-duplex constraint. In a later work [**?**], better protocols were analyzed which bridged the gap between the performance of relays operating in the half-duplex and relays operating in the full-duplex mode. The availability of channel state information (CSI) at the source, destination and relays poses new challenges and opens up possibilities for significant improvements over channels which have CSI at receiver alone. In an earlier work (cf. [3]), expressions for bounds on capacity for single orthogonal relay channels in various situations were derived and power allocation algorithms were obtained for the same. In [1], the authors analyzed a particular protocol in the case of the Gaussian single relay channel, and explored conditions under which the lower and upper bounds on capacity meet. [4] considers the use of multiple relay systems with multiple antennas at each node. In the case where each node has CSI as a receiver only, they show that the optimal strategy is to have independent inputs across antennas at each node.

In this paper, we analyze the simple relay channel (SRC) in which there is CSI at both transmitter and receiver. We obtain and analyze bounds on its capacity and derive a set of conditions under which the bounds meet (thereby characterizing capacity).

II. The Parallel Multiple Relay Channel

The discrete memoryless parallel relay channel is one where each pair of terminals can possibly communicate along two independent links. Figure 1 shows a parallel relay channel with $m = 2$ relays. The input-output pair at each link is referred to, as labelled in the figure. (X_0^{I}, X_0^{II}) refer to the inputs from the source, while $(Y_{m+1}^{I}, Y_{m+1}^{II})$ refer to the outputs at the destination, in channels Phase I and Phase II, respectively. (X_k^{I}, Y_k^{I}) and (X_k^{II}, Y_k^{II}) refer to the input-output pair at the k^{th} relay in channels in Phase I and Phase II, respectively. The input output relations can be expressed in terms of transition probabilities, as shown below:

$$p\left(y^{II}_{(1:m+1)}, y^{I}_{(1:m+1)} | x^{II}_{(0:m)}, x^{I}_{(0:m)}\right) = p\left(y^{I}_{(1:m+1)} | x^{I}_{(0:m)}\right) p\left(y^{II}_{(1:m+1)} | x^{II}_{(0:m)}\right)$$

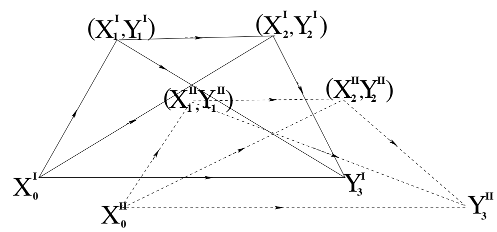

Fig. 1. Parallel relay channel

Theorem 1: A lower bound C_{low} on the capacity of the discrete memoryless parallel multiple relay channel is given by

$$C_{low} = \sup_{\substack{p(x_0^{I} \dots x_m^{I}) \\ p(x_0^{II} \dots x_m^{II})}} \max_{\pi(.)} \min_{1 \le k \le m+1} \{ I(X^{I}_{\pi(0:k-1)}; Y^{I}_{\pi(k)} | X^{I}_{\pi(k:m)}) + I(X^{II}_{\pi(0:k-1)}; Y^{II}_{\pi(k)} | X^{II}_{\pi(k:m)}) \}, \quad (1)$$

where $\pi(.)$ is a permutation on $\{0, \dots, m+1\}$, $\pi(0) = 0$, $\pi(m+1) = m+1$, and $\pi(i;j) = \{\pi(i), \pi(i+1), \dots, \pi(j)\}$.

An upper bound C_{up} on the capacity of the discrete memoryless parallel multiple relay channel is given by

$$C_{up} = \sup_{\substack{p(x_0^{I}, \dots, x_m^{I}) \\ p(x_0^{II}, \dots, x_m^{II})}} \min_{S \subseteq \{1, \dots, m\}} \{ I(X_0^{I} X_S^{I}; Y_{S^C}^{I} Y_{m+1}^{I} | X_{S^C}^{I}) + I(X_0^{II} X_S^{II}; Y_{S^C}^{II} Y_{m+1}^{II} | X_{S^C}^{II}) \} \quad (2)$$

where S^C denotes the complement of S under $\{1, \dots, m\}$.

Proof: The lower bound is derived by applying parallel inputs and outputs to the Degraded-Channel bound given in [5–7]. The upper bound is derived in a similar way, by extending the cut-set bound (cf. [6]). This result when evaluated for a single relay, gives the bounds derived in [1]. ■

III. Simple Multiple Relay Channel - Capacity Characterization

In this section, we analyze the simple relay channel with m relays with each relay and the source having average power constraints. As noted before, a simple relay is a half-duplex relay that is incapable of decoding transmissions from multiple simultaneously transmitting nodes. Accordingly, we consider the simple 2-phase protocol shown in figure 2 wherein the source transmits to the m relays and the intended destination during Phase I of communication and the source and m relays transmit to the destination during Phase II of communication. The parameters $\alpha\,(\bar{\alpha} = 1-\alpha)$ and $\theta\,(\bar{\theta} = 1-\theta)$ denote the fractions of power and bandwidth/time allocated to Phase I (Phase II). We will assume throughout, that there is no delay (or relatively negligible delay) in the transition from Phase I to Phase II.

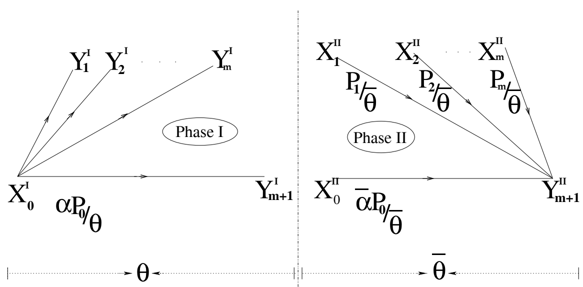

Fig. 2. The Simple Protocol

The input-output relationship for this protocol is summarized in the following equations

$$\text{Phase-I} \quad : \quad Y_k^I = \frac{X_0^I}{\sqrt{d_{0,k}^{\lambda}}} + Z_k^I, \qquad \text{for } \; 1 \leq k \leq m+1.$$

$$\text{Phase-II} \quad : \quad Y_{m+1}^{II} = \sum_{k=0}^{m} \frac{X_k^{II}}{\sqrt{d_{k,m+1}^{\lambda}}} + Z^{II}.$$

where $d_{0,k}$ is the distance between the source and the k^{th} relay and $d_{k,m+1}$ is the distance between the k^{th} relay and the destination while λ is the attenuation co-efficient. Notice that the protocol we described in Figure 2 is just a special case of the parallel relay channel.

We assume that the inputs $\{X_k^{II}\}_{k=1}^m$ are subject to an average power constraints $\{P_k\}_{k=1}^m$, respectively, and the source inputs X_0^I and X_0^{II} are together subject to an average power constraint P_0. The noise terms $\{Z_k^I\}_{k=1}^{m+1}$ are $i.i.d$ Gaussian with distribution $\mathcal{N}(0, \frac{N_k^I}{2})$, while Z^{II} is Gaussian with distribution $\mathcal{N}(0, \frac{N^{II}}{2})$. Let the available bandwidth be W. Let θ be the bandwidth-allocation parameter which gives the fraction of bandwidth (or equivalently time) allotted to Phase I, so that $\overline{\theta}(= 1-\theta)$ indicates the fraction of bandwidth allotted to Phase II of communication. The power-allocation parameter α is such that the transmitter has a power constraint of $(\alpha P_0/\theta)$ during Phase I and $(\overline{\alpha}P_0/\overline{\theta})$ during Phase II, meeting the average power constraint of P_0 at the transmitter. In terms of the normalized parameters $\{\rho_{sr_k} = \frac{P_0}{N_k^I W}\}_{k=1}^m$, $\rho_{sd}^I = \frac{P_0}{N_0^I W}$, $\{\rho_{rd_k} = \frac{P_k}{N^{II} W}\}_{k=1}^m$ and $\rho_{sd}^{II} = \frac{P_0}{N^{II} W}$, we derive upper and lower bounds from equations (1,2) which are stated in the following theorems :

Theorem 2: A lower bound on the capacity in bps/Hz, of a Gaussian simple multiple relay channel with m simple relays (achievable using a decode-forward strategy) is given by

$$C_{low}(\alpha,\theta) \;=\; \min_{1 \leq i \leq m+1} \{C_{i,low}(\alpha,\theta)\} \;, \tag{3}$$

where

$$C_{i,low}(\alpha,\theta) \;=\; \theta C\left(\frac{\alpha \rho_{sr_i}}{\theta\; d_{o,k}^{\lambda}}\right) \qquad \text{for } i \in \{1 \ldots m\}$$

and

$$C_{m+1,low}(\alpha,\theta) \;=\; \theta C\left(\frac{\alpha \rho_{sd}^I}{\theta\; d_{0,m+1}^{\lambda}}\right) + \overline{\theta} C\left(\frac{\overline{\alpha}\frac{\rho_{sd}^{II}}{d_{0,m+1}^{\lambda}} + \sum_{k=1}^m \frac{\rho_{rd_k}}{d_{k,m+1}^{\lambda}}}{\overline{\theta}}\right) \;,$$

with $C(x)$ defined as $\frac{1}{2}log(1+x)$.

An upper bound on the capacity (in bps/Hz) of a Gaussian simple multiple relay channel with m simple relays is given by

$$C_{up}(\alpha,\theta) \;=\; \min_{\pi, 1 \leq i \leq m+1} \left\{C_{\pi(i),up}(\alpha,\theta)\right\} \;, \tag{4}$$

where[1]

$$C_{\pi(i:m),up}(\alpha,\theta) \;=\; \theta C\left(\frac{\alpha\left(\frac{\rho_{sd}^I}{d_{0,m+1}^{\lambda}} + \sum_{k=i}^m \frac{\rho_{sr_{\pi(k)}}}{d_{0,\pi(k)}^{\lambda}}\right)}{\theta}\right) + \overline{\theta} C\left(\frac{\overline{\alpha}\frac{\rho_{sd}^{II}}{d_{0,m+1}^{\lambda}} + \sum_{k=1}^{i-1} \frac{\rho_{rd_{\pi(k)}}}{d_{\pi(k),m+1}^{\lambda}}}{\overline{\theta}}\right) \quad \text{for } i \in \{1 \ldots m\} \;,$$

and

$$C_{m+1,up}(\alpha,\theta) \;=\; \theta C\left(\frac{\alpha \rho_{sd}^I}{\theta\; d_{0,m+1}^{\lambda}}\right) + \overline{\theta} C\left(\frac{\overline{\alpha}\frac{\rho_{sd}^{II}}{d_{0,m+1}^{\lambda}} + \sum_{i=1}^m \frac{\rho_{rd_i}}{d_{i,m+1}^{\lambda}}}{\overline{\theta}}\right) .$$

The following facts need to be noted about the expressions given above.

Fact 1 The input distribution is chosen to be $X_0^I \sim \mathcal{N}(0, \frac{\alpha P_0}{2W\theta})$ at the source during Phase I $X_0^{II} \sim \mathcal{N}(0, \frac{\overline{\alpha}P_0}{2W\overline{\theta}})$ at the source during Phase II and $X_k^{II} \sim \mathcal{N}(0, \frac{P_k}{2W\overline{\theta}})$ independently, at the k^{th} relay during Phase II.

Fact 2 When $\frac{\rho_{sr_k}}{d_{0,k}^{\lambda}} < \frac{\rho_{sd}^I}{d_{0,m+1}^{\lambda}}$ for any $k \in \{1, \ldots m\}$, it can be seen that the lower bound is $C_{k,low}(\alpha,\theta)$, which is less than $\theta C(\frac{\rho_{sd}^I \alpha}{\theta\, d_{0,m+1}^{\lambda}})$. Hence, using a direct transmission by switching the relay off would yield a better rate. This is to be expected since the success of the decode-forward strategy over direct transmission largely depends on the relay's ability to decode messages. This in turn depends on the effective SNR ($\frac{\rho_{sr_k}}{d_{0,k}^{\lambda}}$) being greater than the effective source-destination SNR ($\frac{\rho_{sd}^{I}}{d_{0,m+1}^{\lambda}}$).

Theorem 3: The following conditions are sufficient for the upper and lower bounds to meet.

$$\begin{aligned} \frac{\rho_{sd}^{I}}{d_{0,m+1}^{\lambda}} &\leq \frac{\rho_{sr_i}}{d_{0,i}^{\lambda}} \quad \text{for all} \quad i \in \{1,\ldots,m\} \\ \sum_{i=1}^{m} \frac{\rho_{rd_i}}{d_{i,m+1}^{\lambda}} &\leq \rho_{rd}^{*}\,. \end{aligned} \tag{5}$$

where ρ_{rd}^{*} can be computed numerically by solving the equation below.

$$C_{m+1,low}(\alpha,\theta) \quad \leq \quad C_{i,low}(\alpha,\theta) \quad \forall(\alpha,\theta)$$

for each $i \in \{1,\ldots,m\}$.

Proof: The following equations are valid for all (α,θ).

$$\begin{aligned} C_{m+1,low}(\alpha,\theta) &= C_{m+1,up}(\alpha,\theta) \\ C_{i,low}(\alpha,\theta) &\leq C_{i,up}(\alpha,\theta), \end{aligned} \tag{6}$$

where $C_{i,up}(\alpha,\theta)$ refers to any term $C_{\pi(j:m),up}(\alpha,\theta)$ which contains the term ρ_{sr_i} in its expression.

Now, it can be observed that whenever

$$C_{m+1,low}(\alpha,\theta) \quad \leq \quad C_{i,low}(\alpha,\theta) \quad \forall(\alpha,\theta) \quad \text{for each } i \in \{1,\ldots,m\}, \tag{7}$$

then, by using 6, we can deduce that

$$C_{m+1,up}(\alpha,\theta) = C_{m+1,low}(\alpha,\theta) \leq C_{i,low}(\alpha,\theta) \leq C_{i,up}(\alpha,\theta) \quad \forall i \in \{1,\ldots,m\}.$$

This means that the bounds would meet since

$$\begin{aligned} C_{low}(\alpha,\theta) &= C_{m+1,low}(\alpha,\theta) \\ \text{and} \quad C_{up}(\alpha,\theta) &= C_{m+1,up}(\alpha,\theta). \end{aligned}$$

So the capacity can be explicitly characterized whenever (7) is valid . It can be observed that the following conditions are sufficient to ensure (7) :

$$\begin{aligned} \frac{\rho_{sd}^{I}}{d_{0,m+1}^{\lambda}} &\leq \frac{\rho_{sr_i}}{d_{0,i}^{\lambda}} \quad \text{for all} \quad i \in \{1,\ldots,m\} \\ \sum_{i=1}^{m} \frac{\rho_{rd_i}}{d_{i,m+1}^{\lambda}} &\leq \rho_{rd}^{*}\,. \end{aligned} \tag{8}$$

where ρ_{rd}^{*} can be computed numerically by solving (7). ■

This means that the relays should be far from the destination but shouldn't be farther from the source, than to the destination. Hence, if the relays are located close to the source, then the lower and upper bounds would meet. This is because, the decode-forward strategy works best when the relay is able to decode the messages as against the destination, and the relay is better able to do that, if it is close to the source.

Theorem 4: The expressions for lower and upper bounds ($C_{low}(\alpha,\theta)$ and $C_{up}(\alpha,\theta)$) on the capacity of the simple relay channel are each jointly concave in (α,θ) over $[0,1]\cup[0,1]$. They can hence be optimized over choices of α and θ to obtain a global maxima.

Proof: We will first show that a function $f_{x,z}(\alpha,\theta)$ defined as

$$f_{x,z}(\alpha,\theta) = \theta log\left(1 + \frac{\alpha x + z}{\theta}\right)$$

is jointly concave in α and θ. The Hessian matrix for this function is of the form 2×2 matrix

$$\mathbf{H} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

where

$$\begin{aligned} C_{11} &= \frac{-x^2}{\theta\left(1+\frac{\alpha x+z}{\theta}\right)^2} \\ C_{12} &= \frac{x(\alpha x+z)}{\theta^2\left(1+\frac{\alpha x+z}{\theta}\right)^2} \\ C_{21} &= \frac{x(\alpha x+z)}{\theta^2\left(1+\frac{\alpha x+z}{\theta}\right)^2} \\ C_{22} &= \frac{-(\alpha x+z)^2}{\theta^3\left(1+\frac{\alpha x+z}{\theta}\right)^2} \end{aligned} \tag{9}$$

The determinant of $\mathbf{H}$ is 0, and the eigen values are 0 and $(C_{11}+C_{22})$, so that $\mathbf{H}$ is negative semi-definite. This would mean that the function $f(\alpha,\theta)$ is jointly concave in (α,θ). Since the set $([0,1]U[0,1])$ is a convex set, we can conclude that $f_{x,z}(\alpha,\theta)$ has a global maxima over the set $([0,1]U[0,1])$. Every term in the expression $C_{i,low}(\alpha,\theta)$ can be expressed in terms of $f_{x,z}(\alpha,\theta)$ and $f_{x,z}(\overline{\alpha},\overline{\theta})$. For example,

$$\begin{aligned} C_{m+1,low}(\alpha,\theta) &= f_{x_1,z_1}(\alpha,\theta) + f_{x_2,z_2}(\overline{\alpha},\overline{\theta}) \\ x_1 &= \frac{\rho_{sd}^{I}}{d_{0,m+1}^{\lambda}}\,,\; z_1 = 0 \\ \text{and} \quad x_2 &= \frac{\rho_{sd}^{II}}{d_{0,m+1}^{\lambda}}\,,\; z_2 = \sum_{k=1}^{m} \frac{\rho_{rd_k}}{d_{k,m+1}^{\lambda}} \end{aligned}$$

Since the sum of concave functions is concave, we can conclude that each of the terms $C_{i,low}(\alpha,\theta)$ is jointly concave in (α,θ). Since the point-wise minimum of concave functions yields a concave function (cf. [8]), we have the result for the lower bound. The result for the upper bound is proved in the same way. ■

This theorem proves that the lower bound can be tightened by maximizing over choices of α and θ to obtain

$$C_{low} = \max_{\alpha,\theta}\{C_{low}(\alpha,\theta)\}$$

IV. Numerical Results

Figure 3 shows the increase in achievable rate of the (α, θ)-optimized simple protocol for a simple multiple relay channel with increasing number of relays. In this particular case, the relays were assumed to each have the same power constraint, such that any $\rho_{rd_k} = 20dB$ and each ρ_{sr_k} is varied equally, against which the acheivable rate is plotted. The distances were fixed at unity, and the values of $\rho_{sd}^{II} = \rho_{sd}^{I} = 10dB$.

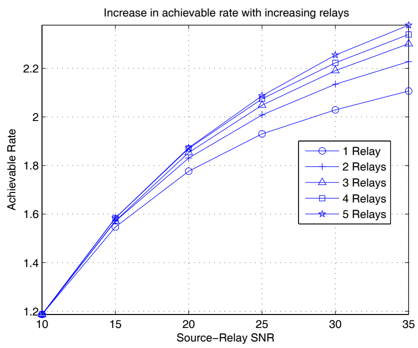

Fig. 3. Increase in achievable rate with increasing relays

In Figure 4, the condition for the equality of the bounds on capacity of the simple multiple relay channel is studied. The possible locations for 2 symmetric relays (placed at same distances from the source and destination), are shown in the picture. It can be seen that the lower and upper bounds meet, when the two relays are positioned closer to the source, than to the destination. In this simulation, $d_{0,3} = 100$, $\rho_{rd_1} = \rho_{rd_2} = 10dB$, $\rho_{sr_1} = \rho_{sr_2} = 5dB$ and $\rho_{sd}^{II} = \rho_{sd}^{I} = 15dB$.

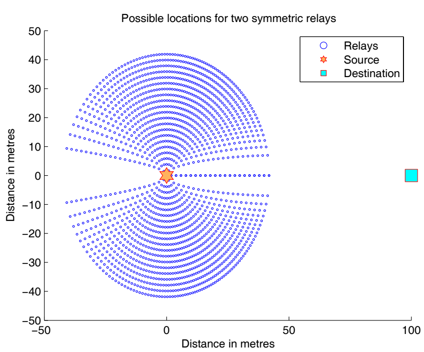

Fig. 4. Location of relays for the bounds to meet

V. Conclusions

In this paper, we analyzed the simple relay channel with multiple relays with CSI at the transmitter, relays and receiver. We derived upper and lower bounds on capacity of this channel and explored conditions under which these bounds meet. It is noted that these conditions are similar to those derived for multiple relays with CSI at receiver only, in [4]. These conditions imply that positioning the relays closer to the transmitter than the receiver will make the two bounds meet. Further, we proved that the lower and upper bounds are jointly-concave in the involved parameters (α, θ), which shows that the achievable rate can be maximized over choices of α and θ. The simple single relay channel which is the same as a channel employing a relay operating in the half-duplex mode and the protocol analyzed here is more general than the one considered in [1] where the source remains silent in the second phase.

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