The Multi-Frequency Small-Signal Model for Buck and Multiphase Interleaving Buck Converters

Yang Qiu, Ming Xu, Kaiwei Yao, Juanjuan Sun, and Fred C. Lee
Center for Power Electronics Systems
Virginia Polytechnic Institute and State University
Blacksburg, VA 24061, USA

Abstract—This paper introduces a multi-frequency small-signal model for buck and multiphase interleaving buck converters. Including the influences from the sideband-frequency components generated by the pulse-width modulation (PWM), this model is applicable beyond half of the switching frequency. In voltage-mode-controlled buck converters, the proposed model predicts the measured phase delay, while the conventional average model fails to explain this phenomenon. With this new model, frequency-domain characteristics are clearly explained for buck and multiphase buck converters. Furthermore, the aliasing effect at half of the switching frequency is examined. Simulation and experimental results are presented to verify the proposed multi-frequency model.

Keywords—small-signal model; multi-frequency model; multiphase buck

I. INTRODUCTION

Future microprocessors will operate at multi-GHz clock frequencies, and will consist of over a billion integrated transistors. Consequently, more than 150-A current at the 0.8-V voltage level will be required [1]. The power-hungry processor poses many stringent challenges on its power supply, the voltage regulator (VR). One special issue is how to meet the requirement of fast transient response with fewer output capacitors. Besides the reasons related to cost, this necessity also exists because of the limited space for VRs in the computer system.

The multiphase interleaving synchronous buck converter, as shown in Fig. 1, is widely adopted in the VR application [2–3]. Many papers have discussed the transient response of multiphase VR and how to improve it [4–9]. It has been shown that the feedback control loop’s bandwidth plays a very important role in the transient response. With a higher bandwidth, fewer output capacitors are needed for the required transient performance [7–9]. Thus, it is advantageous to push the bandwidth as high as possible with a limited switching frequency.

In the past, most of the feedback controller designs have been based on the average model for buck converters. However, because the state-space averaging process eliminates the inherent sampling nature of the switching converter, the accuracy of the average model is questionable at frequencies approaching half of the switching frequency [10]. As the result, the relationship between the control-loop bandwidth and the switching frequency has not been clearly understood.

To predict the sub-harmonic oscillations at half of the switching frequency in peak-current-controlled converters, sampled-data approaches, hybrid approaches, and harmonic-balance approaches have been proposed [10–14]. For the voltage-mode control, it is found that the average model should also be reexamined if a high bandwidth is desired.

As an example, for a 1-MHz single-phase buck with voltage-mode control, Fig. 2 compares the loop gain calculated from the average model with that obtained by using SIMPLIS software. SIMPLIS performs very fast small-signal AC analysis based on a similar scheme as what has been used in the measurements, except that the switching ripples are not considered. Hence the simulated transfer function from SIMPLIS is almost the same as that from measurement.

In Fig. 2, for the case with a 100-kHz bandwidth of the voltage feedback control loop, the average model agrees with the simulation up to half of the switching frequency. However, for a 400-kHz bandwidth design, the average model is good only up to 100 kHz, i.e., one-tenth of the switching frequency. Compared with the average model, the simulation result has a 30° more phase delay at the crossover frequency. This excessive phase drop would result in undesired transient or stability problem if a high-bandwidth converter is designed based on the average model, which cannot predict the high-frequency behaviors.

![Figure 1. An n-phase interleaving buck converter.](image_url)
II. LIMITATIONS OF THE AVERAGE SMALL-SIGNAL MODEL

Fig. 3 demonstrates the hardware setup for measuring the transfer functions of a single-phase buck converter using voltage-mode control. With an inserted sinusoidal voltage source at the swept perturbation frequency, \( f_p \), Fourier analysis is conducted at measured waveforms. After that, the transfer functions are calculated based on the Fourier analysis results. Fig. 4 shows the input and output spectra of the PWM comparator. It is understood that the PWM comparator is a non-linear function. With a perturbation at frequency \( f_p \) inserted at \( V_c \), and the switching frequency, \( f_s \), the output of the PWM comparator, \( V_d \), has infinite frequency components at \( f_p, f_p + f_s, f_p + f_s \), etc. These frequencies other than \( \pm f_p \) are the sidebands around \( f_s \), etc. If the voltage loop is closed, the generated sideband frequency components are fed back to \( V_c \). The fed-back components generate all these frequency components again. Therefore, the sideband effect happens and the frequency components are coupled. However, the traditional average model only includes the \( f_s \) component. If the other frequency components can be ignored, the average model might be good enough. Otherwise, the sideband frequencies should be considered in the model.

Because the feedback control loop (including the power stage and compensator) functions as a low-pass filter, the high-frequency components are attenuated. Considering \( f_s \), lower than the switching frequency, the two most dominant frequencies are \( f_p \) and \( f_p + f_s \). If \( f_p \) is very low and \( f_p + f_s \) is much higher than the bandwidth, the component at \( f_p + f_s \) can be well attenuated by the feedback loop. Fig. 5 shows the simulated voltage waveform at \( V_c \) for a 1-MHz single-phase buck with 100-mV perturbations at 10 kHz. The dominant component is the perturbation frequency.

Although the harmonic-balance technique predicts the phase delays at high frequencies [13–15], it is not straightforward to extract physical meanings out of this complicated model. In order to simplify the modeling, as well as to investigate the control-loop bandwidth limitations and to improve the control designs, it is essential to have a clear picture of the converter characteristics up to the switching frequency. Therefore, this paper introduces the multi-frequency model, which is valid both below and above half of the switching frequency.

Section II reviews the transfer function measurement setup and the limitations of the average model. After that, the concept of multi-frequency modeling is developed in Section III based on a single-phase voltage-mode-controlled buck. Fourier analyses for the PWM comparator [16] are extended to investigate the relationship among different frequency components. In Section IV, this model is applied to the multiphase interleaving buck converter. As a supplementary to the proposed model, the aliasing effect at half of the switching frequency is explored in Section V. Simulation and experimental results are provided to verify the analyses.
Considering the same input and output frequency [16]. It is a pure gain as
\[
G_{PWM}(\omega_p) = \frac{v_d(\omega_p)}{v_r(\omega_p)} = \frac{V_{in}}{V_r},
\]
if \(\omega_p \neq \omega_s/2\). Here, \(V_{in}\) is the input voltage, and \(V_r\) is the peak-to-peak value of the PWM ramp. In order to determine the relationship between these two frequency components, a similar approach as in [16] is applied but with different input and output frequency components. As an example, voltage-mode control with trailing-edge modulation is analyzed. However, the same approach is applicable to average-current model control if the ripple of the feedback signal is small enough. It can also be extended to analyze leading-edge and double-edge modulations.

The input and output waveforms of the trailing-edge PWM comparator are shown in Fig. 8. The control voltage is
\[
v_c(t) = V_c + \hat{v}_c \sin(\omega_s t - \theta),
\]
and the duty ratio for the \(k\)-th cycle is
\[
D_k = \frac{T_k}{T_s} = D + \frac{\hat{v}_c}{V_r} \sin(\omega_s (k-1)T_s + DT_s + (D_k - D)T_s) - \theta,
\]
where \(T_k\) is the on-time of the \(k\)-th cycle, \(T_s\) is the switching period, and
\[
D = \frac{V_c}{V_r}.
\]

Applying the small-signal approximation, it is obtained that
\[
D_k = D + \frac{\hat{v}_c}{V_r} \sin(\omega_s (k-1)T_s - \phi - \theta),
\]
where
\[
\phi = -\omega_s DT_s.
\]

With the definition, the Fourier coefficient for the periodical signal \(v_d\) is expressed as,
\[
v_d(\omega) = \frac{1}{T} \int_{-T/2}^{T/2} v_d(t) e^{-j\omega t} dt.
\]

III. THE MULTI-FREQUENCY SMALL-SIGNAL MODEL

Fig. 7 illustrates the model at the perturbation frequency, \(f_p\), for a single-phase voltage-mode-controlled buck converter. To analyze the small-signal response in the entire frequency region, all the sideband frequency components are included. Therefore, it is named the multi-frequency model. If \(f_p\) is lower than \(f_s\), only one other component at \(f_p \pm f_s\) is important because the converter functions as a low-pass filter. Using Fourier analysis, the PWM transfer function has been derived considering the same input and output frequency [16].
In this paper, it is assumed that
\[
\frac{\omega_p}{\omega_s} = \frac{N}{M},
\]  
(8)
where N and M are positive integers. If the relationship between \(\omega_c\) and \(\omega_1\) cannot be expressed by (8), double integrals are required for the derivation. However, the result is the same so this aspect will not be discussed here for simplicity. Then,
\[
v_d(\omega_p - \omega_s) = \frac{1}{2\pi(N-M)} \int_{0}^{2\pi(N-M)} v_d(t)e^{-j(\omega_p - \omega_s)t} d[(\omega_p - \omega_s)t].
\]  
(9)

Because
\[
v_d(t) = V_{in}. (k-1)T_s < t < (k-1)T_s + T_k, \quad (10)
\]
and
\[
v_d(t) = 0, (k-1)T_s + T_k < t < kT_k, \quad (11)
\]
it can be derived that
\[
v_d(\omega_p - \omega_s) = \frac{V_{in}}{2\pi(N-M)} \sum_{k=1}^{N} \int_{(k-1)\omega_s}^{(k-1)\omega_s + \omega_p} d[(\omega_p - \omega_s)t].
\]  
(12)

If \(\omega_p \neq \omega_s/2\), (12) is rewritten by applying the small-signal approximation as
\[
v_d(\omega_p - \omega_s) = \frac{e^{-j\theta}e^{j2\pi V_{in}/V_r}}{2jV_r}. \quad (13)
\]
The Fourier coefficient for the control signal is
\[
v_c(\omega_p) = \frac{e^{-j\theta}V_c}{2j}; \quad (14)
\]
therefore,
\[
v_d(\omega_p - \omega_s) = \frac{V_{in}}{V_r}e^{j2\pi V_{in}/V_r} \cdot v_c(\omega_p). \quad (15)
\]

With the same approach, it is obtained that at \(\omega_p \neq \omega_s/2\),
\[
v_d(\omega_p) = \frac{V_{in}}{V_r}e^{-j2\pi V_{in}/V_r} \cdot v_c(\omega_p - \omega_s). \quad (16)
\]

With (15) and (16), Fig. 9 shows the multi-frequency model for a single-phase voltage-mode-controlled buck. There are two feedback loops in the model, and each loop represents a certain frequency, but is influenced by the other one. This is the proposed multi-frequency model, and it is valid at \(f_p < f_s\) and \(f_p < f_s/2\). The additional signals added to \(V_d\) represent the influences expressed in (15) and (16).

Using the derived model, it can be calculated that the loop gain at \(f_p\) is
\[
T_s(\omega_p) = \frac{H_s(\omega_p) \cdot G_{LC}(\omega_p) \cdot V_{in}/V_r}{1 + H_s(\omega_p - \omega_s) \cdot G_{LC}(\omega_p - \omega_s) \cdot V_{in}/V_r}, \quad (17)
\]
where \(G_{LC}\) is the transfer function of the power stage LC filter. The numerator is actually the loop gain in the average model. The denominator includes the average-model loop gain at \(f_p f_s\), which reflects the sideband effect, i.e. the influence of the other frequency.

Up to the switching frequency, Fig. 10 compares the loop gain in the average model, in the SIMPLIS simulation and in the multi-frequency model. Fig. 11 shows the measurement result. When \(f_p\) is located in the low-frequency region, \(f_p f_s\) is higher than the voltage-loop bandwidth. The denominator of (17) is approximately equal to one. Therefore, the average model is accurate. When \(f_p\) becomes high and is approaching the switching frequency, \(f_p f_s\) goes into the voltage-loop bandwidth. Under this condition, the denominator of (17) is approximately the loop gain at \(f_p f_s\), which is much higher than one. This high loop gain at \(f_p f_s\) results in a dip around the switching frequency in the loop gain of \(T_s\). Although the average model fails to predict this dip, it does exist in both the experiment and the switching model simulation.

For the same reason, with a low-bandwidth design, the sideband effect has insignificant influence at the crossover frequency, because it is much less than \(f_p/2\). However, when the bandwidth is high, an excessive phase delay is expected at the crossover frequency as the result of sideband effect. As stated in (17), the higher the control bandwidth, the larger impact on the phase margin from the sideband effect. Therefore, the sideband effect limits the possibility of high-bandwidth designs. To help the design at these cases, it is necessary to use the proposed multi-frequency model instead of the conventional average model.
Figure 11. Measured loop gain for the single-phase buck with voltage-mode control.

IV. THE MULTI-FREQUENCY MODEL FOR MULTIPHASE INTERLEAVING BUCK CONVERTERS

To apply this multi-frequency model to the multiphase interleaving buck converter, Fig. 9 needs to be modified. Not only do more paralleled phases appear in the diagram, but also the relationships between the frequency components are changed. For the $m$-th phase in an $n$-phase interleaving buck,

$$\frac{v_{dm}(t) = v_m \cdot (k-1)T_s + (m-1)T_s}{n \cdot t < (k-1)T_s + (m-1)T_s + T_{in}},$$

(18)

and

$$v_{dm}(t) = 0, (k-1)T_s + (m-1)T_s + T_{in} < t < kT_s + (m-1)T_s.$$  

(19)

Through a process similar to that used in the single-phase case, (1) is still correct for the same input and output frequency. For different input and output frequencies, it can be derived that if $\alpha_s = \alpha/2$,

$$v_{dm}(\omega_p - \omega_s) = \frac{V_m}{V_r} \cdot e^{i(2\pi f_s)} \cdot e^{i((m-1)\omega_s)/n} \cdot v_s(\omega_p).$$

Similarly,

$$v_{dm}(\omega_p) = \frac{V_m}{V_r} \cdot e^{-i(2\pi f_s)} \cdot e^{-i((m-1)\omega_s)/n} \cdot v_s(\omega_p - \omega_s).$$

(20)

(21)

Based on (1), (20) and (21), the small-signal model for an $n$-phase interleaving buck at $f_s$ is illustrated in Fig. 12. If $V_r$ and the transfer function $G_{LC}$ are the same for all the phases, the influence from the $f_p f_s$ component on $f_s$ is cancelled at $V_r$. Therefore, the dip that has been observed in Fig. 10 is no longer expected to occur around the switching frequency. In Fig. 13, SIMPLIS simulation is used to verify the multi-frequency model analysis based on a two-phase interleaving buck with voltage-mode control. Compared with that of the single-phase buck, the lowest-frequency dip is around twice the switching frequency. This is because the $f_p \cdot 2f_s$ component as the sideband of $2f_s$ cannot be cancelled.

Figure 12. Multi-frequency model at $f_s$ for an $n$-phase interleaving buck.

Figure 13. Loop gain for a two-phase interleaving buck with voltage control (solid line: SIMPLIS simulation result; dotted line: average-model result).

Generally, for an $n$-phase buck, the first dip appears around $n$-times the switching frequency because of the sideband frequency at $f_s \cdot nf_c$. For example, the loop gain for a four-phase interleaving buck shown in Fig. 14 has the dip around four times the switching frequency. This means that theoretically, for voltage-mode-controlled buck converters, the $n$-phase interleaving technique is capable of increasing the bandwidth to $n$-times that of the single-phase buck.

However, in the loop-gain measurement result shown in Fig. 15 for a two-phase interleaving buck, a small dip exists around the switching frequency although high bandwidth is achieved with sufficient phase margin. This is because in the implementation, the asymmetry of the two phases results in only a partial cancellation of influence from the $f_p f_s$ component. According to Fig. 12, if the two phases’ $V_r$ or $G_{LC}$ are different, or even if the two channels do not have a phase shift of exactly 180°, the influence from the $f_p f_s$ frequency component cannot be perfectly canceled. Therefore, the bandwidth of voltage-mode-controlled multiphase interleaving buck converter is limited in the reality, although it can be increased theoretically. There are risks to push the voltage bandwidth higher when using asymmetric phases. This is why some designers claimed they had achieved bandwidths higher than half the switching frequency, while others claimed they could not.
In this equation, the PWM gain is related to the perturbation's phase. For a single-phase buck, the PWM gain can be derived as

\[ v_{dm}(\omega_s / 2) = \frac{V_d}{V_r} \cdot \left(1 - e^{j(\theta-D)c/2}\right) \cdot v_s(\omega_s / 2). \]  

In this equation, the PWM gain is related to the perturbation's phase \( \theta \). Figure 16 demonstrates the function of \( v_{dm}(\omega_s / 2) \) as a function of the perturbation's relative phase. At certain \( \theta \), the PWM gain has no gain on the perturbation frequency component. While with other phases, the output has higher magnitude.

To verify the results, a simulation with a 1-MHz single-phase buck is performed and shown in Fig. 17. A perturbation is added at the control point with frequency of 500-kHz, i.e., \( f/2 \). The output voltage, \( V_r \), is measured in the frequency domain. With the relative phase of 90°, the output has the largest magnitude of the component at \( f/2 \). With 60° phase, it becomes smaller and there is no \( f/2 \) component at 0°. Because of this aliasing effect, the loop gain has no attenuation for the perturbation with certain phase in the single-phase buck. As aforementioned, the multiphase technique cancels the influence come from the sideband frequencies, so it can improve the control-bandwidth. Hence, it is important to analyze the aliasing effect for multiphase buck case to explore whether there is any cancellation as well.

Different from the single-phase buck, the PWM gain for multiphase buck is related to which phase it is in. For the \( m \)-th phase in an \( n \)-phase interleaving buck,

\[ v_{dm}(\omega_s / 2) = \frac{V_m}{V_r} \left(1 - e^{j(\theta-D)c/2} \cdot e^{j(\theta-D)Nc/2}\right) v_s(\omega_s / 2). \]  

The sum of each phase’s \( V_f \) is calculated to study the aliasing effect for multiphase buck, because the output voltage includes every phase’s influence and symmetrical phases have the same output filter transfer functions. The effective \( V_d \) is

\[ v_{d_{eff}}(\omega_s / 2) = \sum_{m=1}^{n} v_{dm}(\omega_s / 2) = \frac{Nv_m}{V_r} v_s(\omega_s / 2). \]  

Therefore, the total effect is represented by a constant gain, which means that the aliasing effect is cancelled at \( f/2 \) for multiphase buck.

The simulation result with a 1-MHz two-phase buck is shown in Fig. 18. Unlike the single-phase case, the output voltage has the same spectrum for perturbations with different phases, which means the aliasing effect is canceled at \( f/2 \). However, similar to the sideband effect, the cancellation is dependent on the symmetry among phases. As the result, when the bandwidth is higher than half of the switching frequency, it is necessary to consider the parameter tolerance and the aliasing effect that cannot be completely cancelled.
The conventional average model fails to predict the phase delays even inside the control-loop bandwidth for high-bandwidth designs in voltage-mode-controlled buck. The influences from the sideband frequency components, which are generated by the PWM comparator, should be considered. To predict the system behavior and improve the control design, this paper introduces the multi-frequency model, which is valid above half the switching frequency. Models are developed for both the single-phase buck and the multiphase interleaving buck. As a supplementary, the aliasing effect at half of the switching frequency is also discussed. Simulation and experimental results have verified the proposed model and corresponding analyses. For voltage-mode control, the multiphase interleaving technique cancels the sideband effect and the aliasing effect. Therefore, theoretically the control-loop bandwidth can be pushed higher than half the switching frequency. However, the asymmetry among phases results in design risks to push the control-loop bandwidth in implementations.

VI. CONCLUSION

ACKNOWLEDGMENT

This work was conducted with the use of SIMPLIS software, donated in kind by Transim Technology of the CPES Industrial Consortium.

REFERENCES