1. (a) Prove the following theorem.

**Theorem 0.1** If \( \{ A, b \} \) is controllable there is a row vector \( c \) that makes \( \{ A, c \} \) observable.

If \( A, b \) is controllable, there is a square invertible matrix \( T \) such that \( A_c = T^{-1} AT, \ b_c = T^{-1} b \) is in controller canonical form. If

\[
c_c = \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix}
\]

then we know

\[
c_c(sI - A_c)^{-1}b_c = \frac{b_1s^{n-1} + b_2s^{n-2} + \cdots + b_n}{(s - \lambda_1)(s - \lambda_2) \cdots (s - \lambda_n)}
\]

where the \( \lambda_i \) are the eigenvalues of \( A_c \) (and of \( A \)). Choose the \( b_i \) to give a numerator polynomial whose roots do not lie at any of the eigenvalues \( \lambda_i \). Then there is no cancellation between numerator and denominator and so \( \{ A_c, b_c, c_c \} \) is minimal thus observable. It then follows that \( c = c_c T^{-1} \) makes \( \{ A, c \} \) observable.

(b) Either prove the following conjecture or find a counter-example

**Conjecture 0.2** Given any square matrix \( A \) there is a row vector \( c \) that makes \( \{ A, c \} \) observable.

This is FALSE. Take the 2 \( \times \) 2 example

\[
A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

and then \( O = \begin{bmatrix} c \\ c \end{bmatrix} \) which is clearly singular no matter what \( c \) you choose. Thus no row-vector \( c \) can make \( \{ A, c \} \) an observable pair.
2. Consider the realization

\[
A = \begin{bmatrix}
1 & 0 & 0 \\
-1 & -1 & 0 \\
0 & 0 & 2
\end{bmatrix} \quad b = \begin{bmatrix}
4 \\
0 \\
0
\end{bmatrix} \quad c = \begin{bmatrix}
1 & 2 & -1
\end{bmatrix}
\]

(a) The eigenvalues of \( A \) are obviously 1, \(-1\), and 2. Please complete the following table by placing “Y” (for “YES”) or an "N" (for “NO”). In the blank entries.

<table>
<thead>
<tr>
<th>eigenvalue</th>
<th>Controllable?</th>
<th>Observable?</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda = 1 )</td>
<td>Y</td>
<td>N</td>
</tr>
<tr>
<td>( \lambda = -1 )</td>
<td>Y</td>
<td>Y</td>
</tr>
<tr>
<td>( \lambda = 2 )</td>
<td>N</td>
<td>Y</td>
</tr>
</tbody>
</table>

(b) If \( g(s) = c(sI - A)^{-1}b \) and I tell you that \( g(0) = 4 \). Can you find \( g(s) \) without computing any entries in \( (sI - A)^{-1} \)? Explain.

Since the \( \lambda = 2 \) eigenvalue is uncontrollable, it will cancel. Likewise, the \( \lambda = 1 \) eigenvalue, being unobservable will cancel, leaving a denominator of \( (s + 1) \).

So if \( c(sI - A)^{-1}b = \frac{n(s)}{d(s)} \) we know \( d(s) = (s + 1) \) and \( n(s) \) is constant. So, we conclude

\[
g(s) = \frac{4}{(s + 1)}
\]
3. A given realization \( \{ A, b, c \} \) is known to be controllable, to have \( A \in \mathbb{R}^{5 \times 5} \), to have an observable subspace spanned by
\[
\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}
\]
and to have distinct (non-repeated) eigenvalues for \( A \).

Please mark the following assertions as “T” (must be true for this \( \{ A, b, c \} \)), “F” (must be false for this \( \{ A, b, c \} \)), or “NI” (not enough information is given/known about this \( \{ A, b, c \} \) to decide T/F).

(a) Realization is asymptotically stable (i.e. all of \( A \)'s eigenvalues have negative real part). NI
There is just no way to tell.

(b) \( c(sI - A)^{-1}b \) will have two poles after cancelling all common factors between numerator and denominator. T
\( \dim(X_o) = 2 \) so \( \dim(X_o) = 3 \) and three eigenvalues will cancel out of the five. There are no cancellations due to uncontrollability, so this leaves two eigenvalues showing up as poles.

(c) The fifth entry in \( c \) is zero. T
\[
X_o = X_o^\perp = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}
\]
Since \( A \) has distinct eigenvalues, there must be three independent (right) eigenvectors, \( v_1, v_2, v_3 \) in this span all of which satisfy \( cv_i = 0 \) (Corresponding to the three unobservable modes). It thus follows that for some constants \( \alpha_1, \alpha_2, \alpha_3 \), not all zero that
\[
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 \Rightarrow c = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \alpha_1 cv_1 + \alpha_2 cv_2 + \alpha_3 cv_3 = 0
\]
So the fifth entry of \( c \) is zero.

(d) If the system is started, with no input, from an initial condition
\[
x(0) = \begin{bmatrix} x_{10} \\ x_{20} \\ x_{30} \\ x_{40} \\ x_{50} \end{bmatrix}
\]
then from observations of \( y(t) \) a unique value for \( x_{40} \) can be determined. F
Any two initial conditions \( x(0) \) and \( \bar{x}(0) \) that differ by something in \( X_o \) cannot be distinguished. We see that some vectors in \( X_o \) have non-zero 4th entry. So \( x_{40} \) cannot be uniquely determined.
(e) The vector
\[
\begin{bmatrix}
0 \\
0 \\
0 \\
1
\end{bmatrix}
\]
is an (right) eigenvector of \(A\). Not known. Can anyone solve this?

4. MATLAB useful
Return to the spring-mass damper system of HW#2 which had the realization
\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 \\
-\frac{K}{M} & 0 & \frac{K}{M} & 0 \\
0 & 0 & 0 & 1 \\
\frac{K}{m} & 0 & -\frac{K}{m} & 0
\end{bmatrix}; \quad b = \begin{bmatrix}
\frac{1}{M} \\
0
\end{bmatrix}; \quad c = \begin{bmatrix}
1 & 0 & 0 & 0
\end{bmatrix}; \quad d = 0
\]

Suppose \(K = .1\), \(M = 2\), and \(m = .2\).

(a) Find the open-loop poles and zeros.

The transfer function is
\[
G(s) = \frac{s^2 + \frac{K}{m}}{s^2 (M+Mm) + 2\frac{s^2 + 1/2}{s^2 (4s^2 + .22)}} = 2 \frac{10s^2 + 5}{s^2 (40s^2 + 22)}
\]

The open loop zeros are at \(s = \pm j\sqrt{1/2}\). The open loop poles are \(s = 0, 0, \pm j\sqrt{11/20}\)

(b) Find the state-feedback gain vector, \(K\), that places the eigenvalues of \(A-bK\) at \(-1, -2, -3, -4\).

\(K = \begin{bmatrix}
68.9 \\
20 \\
27.1 \\
180
\end{bmatrix}\)

(c) For the cost function
\[
J(x_0, u) = \int_0^\infty y^2(t) + 2u^2(t) \, dt
\]
find the LQR optimal state feedback gain and the optimal closed-loop poles.

By Chang-Letov, the optimal closed loop poles are the left half-plane roots of
\[
1 + \frac{1}{2} \left[ 2 \frac{10s^2 + 5}{s^2 (40s^2 + 22)} \right]^2
\]
since for this system \(G(-s) = G(s)\). These are the left half-plane solutions of
\[
s^4 \left[ (40s^2 + 22) \right]^2 + \frac{4}{2} \left[ 10s^2 + 5 \right]^2 = 0
\]
which are \(s = -0.3970 \pm j0.4151, -0.0164 \pm j0.7318\). We can now compute the state feedback gain \(K\) that will place the eigenvalues of \(A-bK\) at these locations. The result is
\(K = \begin{bmatrix}
0.6834 \\
1.6534 \\
0.0237 \\
0.0914
\end{bmatrix}\)