Second Order RLC Filters

1 RLC Lowpass Filter

A passive RLC lowpass filter (LPF) circuit is shown in the following schematic.

\[
\begin{align*}
\text{RLC Lowpass Filter} \\
\begin{array}{c}
\text{Input}\ v_S(t) \\
\downarrow \\
\text{RC}\ \\
\downarrow \\
\text{Output}\ v_O(t)
\end{array}
\end{align*}
\]

Using phasor analysis, \( v_O(t) \leftrightarrow V_O \) is computed as

\[
V_O = \frac{1}{j\omega C} V_S = \frac{1}{LC} \frac{1}{(j\omega)^2 + j\omega \frac{R}{L} + \frac{1}{LC}} V_S.
\]

Setting \( \omega_0 = 1/\sqrt{LC} \) and \( 2\zeta \omega_0 = R/L \), where \( \omega_0 \) is the (undamped) natural frequency and \( \zeta \) is the damping ratio, yields

\[
V_O = \frac{\omega_0^2}{(j\omega)^2 + j\omega 2\zeta \omega_0 + \omega_0^2} V_S \quad \leftrightarrow \quad v_O^{(2)}(t) + 2\zeta \omega_0 v_O^{(1)}(t) + \omega_0^2 v_O(t) = \omega_0^2 v_S(t).
\]

The blockdiagram that represents this differential equation is

Unit Step Response. By definition, the unit step response \( g(t) \) of a circuit is the zero-state response (ZSR) to the input \( v_s(t) = u(t) \). For the 2'nd order LPF considered here the unit step response is of the form (if \( \zeta \neq 1 \))

\[
g(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t} + 1, \quad t \geq 0 \quad \implies \quad g^{(1)}(t) = s_1 K_1 e^{s_1 t} + s_2 K_2 e^{s_2 t}, \quad t \geq 0,
\]
with initial conditions $g(0) = 0$ and $g^{(1)}(0) = 0$. The values of $s_1$ and $s_2$ are the solutions of the characteristic equation

$$s^2 + 2\zeta\omega_0 s + \omega_0^2 = 0 \implies s_{1,2} = (-\zeta \pm \sqrt{\zeta^2 - 1})\omega_0,$$

and the properties of $g(t)$ change fundamentally depending on whether $\zeta > 1$, $\zeta = 1$, or $\zeta < 1$.

**Overdamped Case, $\zeta > 1$.** In this case the characteristic equation has two real solutions

$$s_1 = -\alpha_1, \quad \alpha_1 = (\zeta - \sqrt{\zeta^2 - 1})\omega_0, \quad \text{and} \quad s_2 = -\alpha_2, \quad \alpha_2 = (\zeta + \sqrt{\zeta^2 - 1})\omega_0.$$

Note that $\alpha_1 < \alpha_2$. The unit step response is of the form

$$g(t) = K_1 e^{-\alpha_1 t} + K_2 e^{-\alpha_2 t} + 1, \quad t \geq 0 \implies g^{(1)}(t) = -\alpha_1 K_1 e^{-\alpha_1 t} - \alpha_2 K_2 e^{-\alpha_2 t}, \quad t \geq 0.$$

Using initial conditions $g(0) = 0$ and $g^{(1)}(0) = 0$ yields

$$K_1 + K_2 = -1, \quad \alpha_1 K_1 + \alpha_2 K_2 = 0. \implies \begin{bmatrix} 1 & 1 \\ \alpha_1 & \alpha_2 \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

From this $K_1$ and $K_2$ are obtained as

$$K_1 = \frac{-\alpha_2}{\alpha_2 - \alpha_1}, \quad K_2 = \frac{\alpha_1}{\alpha_2 - \alpha_1},$$

and thus the unit step response for a 2'nd order overdamped LPF is

$$g(t) = 1 - \frac{\alpha_2 e^{-\alpha_1 t} - \alpha_1 e^{-\alpha_2 t}}{\alpha_2 - \alpha_1}, \quad t \geq 0.$$

Note that

$$g^{(1)}(t) = \frac{\alpha_1 \alpha_2}{\alpha_2 - \alpha_1} (e^{-\alpha_1 t} - e^{-\alpha_2 t}), \quad t \geq 0,$$

and therefore $g^{(1)}(t) = 0$ requires that $e^{-\alpha_1 t} = e^{-\alpha_2 t}$ which can only happen at $t = 0$ or $t = \infty$ if $\alpha_1 \neq \alpha_2$. This implies that the extrema of $g(t)$ occur at $t = 0$ (where $g(0) = 0$) and at $t = \infty$ (where $g(\infty) = 1$) and thus $g(t)$ has no overshoot.

**Critically Damped Case, $\zeta = 1$.** In this case the characteristic equation has one real double solution

$$s_1 = s_2 = -\alpha, \quad \alpha = \omega_0,$$

and the unit step response is of the form

$$g(t) = K_1 e^{-\alpha t} + K_2 t e^{-\alpha t} + 1, \quad t \geq 0,$$

$$\implies g^{(1)}(t) = -\alpha K_1 e^{-\alpha t} + K_2 e^{-\alpha t} - \alpha K_2 t e^{-\alpha t}, \quad t \geq 0.$$

Using initial conditions $g(0) = 0$ and $g^{(1)}(0) = 0$ yields

$$K_1 = -1, \quad K_2 = -\alpha,$$
and thus the unit step response for a 2’nd order critically damped LPF is

\[ g(t) = 1 - (1 + \alpha t) e^{-\alpha t}, \quad t \geq 0. \]

Note that

\[ g^{(1)}(t) = \alpha^2 t e^{-\alpha t}, \quad t \geq 0, \]

and therefore \( g^{(1)}(t) = 0 \) requires either \( t = 0 \) or \( t = \infty \), which implies that the extrema of \( g(t) \) are 0 (at \( t = 0 \)) and 1 (at \( t = \infty \)) and thus \( g(t) \) has no overshoot.

**Underdamped Case, \( \zeta < 1 \).** In this case the characteristic equation has two complex solutions which are conjugates of each other

\[ s_1 = -\alpha + j\beta, \quad \text{and} \quad s_2 = s_1^* = -\alpha - j\beta, \quad \text{where} \quad \alpha = \zeta \omega_0, \quad \beta = \sqrt{1 - \zeta^2} \omega_0. \]

The unit step response is of the form

\[ g(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t} + 1, \quad t \geq 0 \quad \implies \quad g^{(1)}(t) = s_1 K_1 e^{s_1 t} + s_2 K_2 e^{s_2 t}, \quad t \geq 0. \]

Substituting \( s_{1,2} = -\alpha \pm j\beta \) and using initial conditions \( g(0) = 0 \) and \( g^{(1)}(0) = 0 \) yields

\[ K_1 + K_2 = -1, \quad \implies \quad \begin{bmatrix} 1 & 1 \\ -\alpha + j\beta & -\alpha - j\beta \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}. \]

From this \( K_1 \) and \( K_2 = K_1^* \) are obtained as

\[ K_1 = -\frac{\beta + j\alpha}{2\beta} = \rho e^{j\phi}, \quad K_2 = -\frac{\beta - j\alpha}{2\beta} = \rho e^{-j\phi}, \]

where

\[ \rho = \frac{\sqrt{\alpha^2 + \beta^2}}{2\beta} = \frac{1}{2 \sqrt{1 - \zeta^2}}, \quad \text{and} \quad \phi = \pi - \tan^{-1} \frac{\alpha}{\beta} = \pi - \tan^{-1} \frac{\zeta}{\sqrt{1 - \zeta^2}}. \]

Thus, the unit step response for a 2’nd order underdamped LPF is

\[ g(t) = 1 + \rho e^{-\alpha t} \left( e^{j(\beta t + \phi)} + e^{-j(\beta t + \phi)} \right) = 1 + 2\rho e^{-\alpha t} \cos(\beta t + \phi), \quad t \geq 0, \]

or, with \( \rho \) and \( \phi \) substituted

\[ g(t) = 1 - \frac{e^{-\alpha t}}{\sqrt{1 - \zeta^2}} \cos \left( \beta t - \tan^{-1} \frac{\zeta}{\sqrt{1 - \zeta^2}} \right), \quad t \geq 0. \]

To obtain a formula for \( g^{(1)}(t) \) easily, first note that \( s_2 K_2 = (s_1 K_1)^* \) and

\[ s_1 K_1 = (-\alpha + j\beta) \frac{-\beta + j\alpha}{2\beta} = \frac{\alpha^2 + \beta^2}{2j\beta}. \]
Then use \( g^{(1)}(t) = (s_1 K_1 e^{j\beta t} + s_2 K_2 e^{-j\beta t}) e^{-\alpha t} \) to obtain

\[
g^{(1)}(t) = \frac{\alpha^2 + \beta^2}{\beta} e^{-\alpha t} \left(\frac{e^{j\beta t} - e^{-j\beta t}}{2j}\right) = \frac{\alpha^2 + \beta^2}{\beta} e^{-\alpha t} \sin \beta t, \quad t \geq 0.
\]

To find the times where the extrema of \( g(t) \) occur, set \( g^{(1)}(t) = 0 \) and solve for \( t \). The sine has zero crossings for \( \beta t = k\pi \) and, for \( k = 0 \), \( g(t) = g(0) = 0 \) clearly has a minimum. The largest and most interesting maximum (due to underdamping) of \( g(t) \) happens when \( bt = \pi \Rightarrow t = \pi/\beta \). The value of the maximum is computed as

\[
g_{\text{max}} = g\left(\frac{\pi}{\beta}\right) = 1 + 2\rho e^{-\pi\alpha/\beta} \cos(\pi + \phi) = 1 + \frac{e^{-\pi\zeta/\sqrt{1-\zeta^2}}}{\sqrt{1-\zeta^2}} \cos \left( \tan^{-1} \frac{\zeta}{\sqrt{1-\zeta^2}} \right).
\]

Using the identity

\[
\cos \left( \tan^{-1} \frac{x}{\sqrt{1-x^2}} \right) = \sqrt{1-x^2},
\]

the final result is

\[
g_{\text{max}} = 1 + e^{-\pi\zeta/\sqrt{1-\zeta^2}} \quad \Rightarrow \quad \text{Overshoot in \%: } 100 e^{-\pi\zeta/\sqrt{1-\zeta^2}}.
\]

The following graph shows the unit step response \( g(t) \) for several values of \( \zeta \).
**Frequency Response.** From the phasor analysis the system function of the LPF is obtained as

\[ H = \frac{V_O}{V_S} = \frac{\omega_0^2}{\omega_0^2 - \omega^2 + j 2\zeta\omega_0 \omega} . \]

The magnitude and the phase of \( H \) are

\[ |H| = \frac{\omega_0^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\zeta\omega_0 \omega)^2}} , \quad \text{and} \quad \angle H = -\tan^{-1} \frac{2\zeta\omega_0 \omega}{\omega_0^2 - \omega^2} . \]

Note that, at \( \omega = \omega_0 \),

\[ |H|_{\omega_0} = \frac{1}{2\zeta} , \quad \text{and} \quad \angle H_{\omega_0} = -90^\circ , \]

and thus \( \zeta \) and \( \omega_0 \) can be easily determined from the magnitude and phase of \( H \).

**2 RLC Bandpass Filter**

A passive \( RLC \) bandpass filter (BPF) circuit is shown in the following schematic.

Using phasor analysis, \( v_O(t) \leftrightarrow V_O \) is computed as

\[ V_O = \frac{j\omega L}{R + \frac{1}{j\omega RC + 1}} V_S = \frac{j\omega L}{(j\omega)^2 + j\omega RC + 1} V_S . \]

Setting \( \omega_0 = 1/\sqrt{LC} \) and \( 2\zeta\omega_0 = 1/(RC) \) yields

\[ V_O = \frac{j\omega 2\zeta\omega_0}{(j\omega)^2 + j\omega 2\zeta\omega_0 + \omega_0^2} V_S \quad \iff \quad v_O^{(2)}(t) + 2\zeta\omega_0 v_O^{(1)}(t) + \omega_0^2 v_O(t) = 2\zeta\omega_0 v_S^{(1)}(t) . \]

The blockdiagram that represents this differential equation is (after integrating both sides so that the input is \( v_S(t) \) rather than \( v_S^{(1)}(t) \))
**Frequency Response.** From the phasor analysis the system function of the BPF is obtained as

\[ H = \frac{V_O}{V_S} = \frac{j 2\zeta \omega_0 \omega}{\omega_0^2 - \omega^2 + j 2\zeta \omega_0 \omega}. \]

The magnitude and the phase of \( H \) are

\[ |H| = \frac{2\zeta \omega_0 \omega}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\zeta \omega_0 \omega)^2}}, \quad \text{and} \quad \angle H = \pi/2 - \tan^{-1} \frac{2\zeta \omega_0 \omega}{\omega_0^2 - \omega^2}. \]

Note that, at \( \omega = \omega_0 \) (the center frequency of the BPF),

\[ |H|_{\omega_0} = 1, \quad \text{and} \quad \angle H_{\omega_0} = 0^\circ. \]

To obtain the lower half-power (or -3dB) frequency \( \omega_3^- \) of the BPF, set

\[ \omega_0^2 - \omega_3^2 = 2\zeta \omega_0 \omega_3^- \quad \Rightarrow \quad \omega_3^- = (\sqrt{1 + \zeta^2} - \zeta) \omega_0. \]

Similarly, the upper half-power (or -3dB) frequency \( \omega_3^+ \) of the BPF is obtained from

\[ \omega_3^2 + \omega_0^2 = 2\zeta \omega_0 \omega_3^+ \quad \Rightarrow \quad \omega_3^+ = (\sqrt{1 + \zeta^2} + \zeta) \omega_0. \]

Thus, the half-power (or -3dB) bandwidth of the BPF is equal to \( 2\zeta \omega_0 \), and \( \zeta \) and \( \omega_0 \) can therefore be determined from the magnitude and phase of \( H \).