Gaussian beams

Solution of scalar paraxial wave equation (Helmholtz equation) is a Gaussian beam, given by:

\[
E(\vec{r}) = A_0 \frac{w_0}{w(z)} e^{-\frac{\rho^2}{w^2(z)}} e^{-jkz - jk\frac{\rho^2}{2R(z)}} + j\zeta(z)
\]

where

\[
\begin{align*}
\text{Size} & \quad w(z) = w_0 \sqrt{1 + \left(\frac{z}{z_0}\right)^2} \\
\text{Radius of curvature} & \quad R(z) = z \left[1 + \left(\frac{z_0}{z}\right)^2\right] \\
\text{Gouy phase} & \quad \zeta(z) = \tan^{-1}\left(\frac{z}{z_0}\right)
\end{align*}
\]

Note that \( R(z) \) does not obey ray tracing sign convention. Unfortunately there’s no particularly good way to fix this.
Gaussian beams

Detailed view

Real part of $E$ vs. radius and $z$

At $z = z_0$,

$$I(0,z_0) = I(0,0)/2 \quad w(z_0) = \sqrt{2}w_0 \quad R(z_0) = 2z_0 \text{ (min value)}$$

Any constant times $w(z)$ is a ray path. Note that the rays are converging and diverging spherical wave except near the focus where they bend. Ergo rays do not always travel in straight lines – the region near the focus violates the slowly-varying envelope approximation.

Conversion formulas

$$w_0 = \sqrt{\frac{\lambda}{\pi}} z_0 = \frac{\lambda}{\pi} \frac{1}{\theta_0} \quad \theta_0 = \frac{\lambda}{\pi} \frac{1}{w_0} = \sqrt{\frac{\lambda}{\pi}} \frac{1}{z_0} \quad z_0 = \frac{\lambda}{\pi} \theta_0^{-2} = \frac{\pi}{\lambda} w_0^2 = \frac{w_0}{\theta_0}$$
Gaussian beam parameter \( q(z) \)

The complete Gaussian beam expression normalized to intensity

\[
E(\vec{r}) = A_0 \left( \frac{1}{\sqrt{\pi} w_0} \right)^D e^{-\frac{\rho^2}{w^2(z)}} e^{\frac{-jk\rho^2}{2R(z)} + j\zeta(z)}
\]

\( D \) is number of transverse dimensions = 1, 2

Define the complex radius of curvature

\[
\frac{1}{q(z)} = \frac{1}{R(z)} - j \frac{\lambda}{\pi w^2(z)}
\]

What is \( q(z) \)?

\[
q(z) = \frac{1}{\left[ 1 + \left( \frac{z_0}{z} \right)^2 \right] - j \left[ 1 + \left( \frac{z}{z_0} \right)^2 \right]} = \left[ \frac{z - jz_0}{z^2 + z_0^2} \right]^{-1} = z + jz_0
\]

\[
\arg(q) = \tan^{-1} \frac{z_0}{z}
\]

\[
|q| = \sqrt{1 + \left( \frac{z}{z_0} \right)^2} = \frac{w(z)}{w_0}
\]

Can now write Gaussian above as

\[
E(\vec{r}) = j A_0 \left( \frac{1}{\sqrt{\pi} w_0} \right)^D e^{\frac{z_0}{q(z)} e^{-\frac{jk\rho^2}{2q(z)} - jkz}}
\]

Note that phase of \( j/q(z) \) is \( \zeta \).
How does q change with transfer and refraction?

Free space:

\[ q_1 = z_1 + jz_0 \]
\[ q_2 = z_2 + jz_0 = q_1 + (z_2 - z_1) \]

so \( q_2 = q_1 + \Delta z \)

Thin lens

\[ \frac{1}{q(z)} = \frac{1}{R(z)} - j \frac{\lambda}{\pi w^2(z)} \]

Start with expression for \( 1/q(z) \)

\[ \frac{1}{R'} = \frac{1}{R} - \frac{1}{f} \]

Thin lens equation expressed as change in curvature of wave

NOTE HOW GAUSSIAN BEAM SIGN CONVENTION HAS CHANGED THE SIGN

\[ \frac{1}{q'} = \frac{1}{q} - \frac{1}{f} \]

Apply to \( 1/q \)

\[ q' = \frac{q}{-q/f + 1} \]

Solve for q
ABCĐ approach to q

Remember the ABCĐ matrices for thin lens refraction and free-space transfer

\[
R_k = \begin{bmatrix}
1 & 0 \\
-\phi_k & 1
\end{bmatrix} \quad T_k = \begin{bmatrix}
1 & t'_k \\
0 & 1
\end{bmatrix}
\]

and define the evolution equation for q

\[
q' = \frac{Aq + B}{Cq + D}
\]

Check for free space:

\[
q' = \frac{1q + t'_k}{0q + 1} = q + t'_k
\]

Check for thin lens:

\[
q' = \frac{1q + 0}{-\phi_k q + 1} = \frac{q}{-q/f + 1}
\]

Which says, rather remarkably, that we can model the propagation of a Gaussian beam through a paraxial optical system using ray matrices. But there’s a better way to do so (at least I like it’s better).
Representation of Gaussian beams by complex rays (1)

Define the following three rays. Note their suggestive names and relationship to the Gaussian beam.

- **Waist ray** \( \Omega \)
- **Divergence ray** \( \Delta \)
- **Chief ray** \( \theta_0 \)

**Paraxial ray trajectory form**

\[
\begin{align*}
\Omega(z) &= w_0 \\
\Delta(z) &= z \theta_0
\end{align*}
\]

**ABCD vector form**

\[
\begin{align*}
\Omega_0 &= \begin{bmatrix} w_0 \\ 0 \end{bmatrix} \\
\Delta_0 &= \begin{bmatrix} 0 \\ \theta_0 \end{bmatrix}
\end{align*}
\]

Define the complex ray trajectory

\[
\Gamma(z) = \Delta(z) + j\Omega(z)
\]

This is Greynolds’ definition and yields the proper form of \( q \). Arnaud’s definition yields \( q^* \).

You can then show that this ray contains \( q(z) \)

\[
\frac{\Gamma(z)}{\Delta(z)} = \begin{bmatrix} y_\Delta + jy_\Omega \\ u_\Delta + ju_\Omega \end{bmatrix}
\]

Ray heights over ray slopes

\[
= \frac{z\theta_0 + jw_0}{\theta_0} = z + jz_0 = q(z)
\]

E.g. at \( z=0 \)

J. Arnaud, Applied Optics, V24, N4, p. 538, 15 Feb 1985
A. W. Greynolds, SPIE V 560, p. 33, 1985
M&M A2.5
Representation of Gaussian beams by complex rays (2)

First we note:

\[ \left( y_\Omega u_\Delta - y_\Delta u_\Omega \right) \frac{n}{n'} = \frac{\lambda'}{\pi} \]

Lagrange invariant \((N_{\text{spots}} = 1)\)

By brute force tracing of the rays, we can find the following Gaussian parameters based on the two rays at that point:

\[ \theta_0 = \sqrt{u_\Delta^2 + u_\Omega^2} \]

Which gives all other beam parameters

\[ w(z) = \sqrt{y_\Omega^2(z) + y_\Delta^2(z)} \]

1/e field radius at this \(z\)

We could use these two and the expressions for the Gaussian beam parameters to generate the complete Gaussian, but this would be a bit tedious. A more elegant way is to use the complex ray formalism:

\[ q(z) = \frac{\Gamma(z)}{d\Gamma/dz} \]

\[ = \frac{y_\Delta + zu_\Delta + j(y_\Omega + zu_\Omega)}{u_\Delta + ju_\Omega} \]

At plane \(z \neq 0\)

Which, apart from the on-axis phase \(k_0S\) gives the full Gaussian beam at this plane \(z\).
Representation of Gaussian beams by complex rays (3)

On-axis examples:

1) \( \lambda = 1 \, \mu m, \quad w_0 = 10 \lambda, \quad f = 500 \, \mu m, \quad 1\ f - 1\ f \) system.

2) \( \lambda = 1 \, \mu m, \quad w_0 = 10 \lambda, \quad f = 500 \, \mu m, \quad 2\ f - 2\ f \) system.

3) \( \lambda = 1 \, \mu m, \quad w_0 = 10 \lambda, \quad f = 500 \, \mu m, \quad 3\ f - 3/2\ f \) system.

Notes
- In (1), second waist is at Fourier plane, as expected.
- In (2), second waist occurs before image plane, as expected.
- In (3), as distance to lens increases, waist moves to paraxial image plane.
Representation of Gaussian beams by complex rays (4)

Off-axis examples:

1) $\lambda = 1 \mu m$, $w_0 = 10 \lambda$, $f = 500 \mu m$, 1 f – 1 f system.

2) $\lambda = 1 \mu m$, $w_0 = 10 \lambda$, $f = 500 \mu m$, 2 f – 2 f system.

3) $\lambda = 1 \mu m$, $w_0 = 10 \lambda$, $f = 500 \mu m$, 3 f – 3/2 f system.

Notes

- In (1), waist is centered at zero (as expected of FT geometry)
- In (2), image is at -10 \mu m, expected from M=-1.
- This type of problem is not possible with the ABCD formalism.
Do Gaussian beams obey paraxial imaging? (1/3)

Answer: Yes. The image is also a Gaussian E field distribution in amplitude and any point on the object down from the peak by some value, say 1/e for the point \( w(z) \), will image to the point on the image down from the peak by the same value. Shown above only for real objects conjugate to real images ( \(-t \geq f\) ).
Do Gaussian beams obey paraxial imaging? (2/3)

How about for virtual objects?

Answer: Still works.
Do Gaussian beams obey paraxial imaging? (3/3)

How about for virtual images?

Conclusion: All parts of the object Gaussian image correctly to the appropriate parts of the image Gaussian including both real and virtual objects and images.

Corollary: If you apply paraxial imaging to the object Gaussian over all $z$, you generate the image Gaussian over all $z$. Gaussian beams obey paraxial imaging exactly.
Design example

**Collimator lens**

\[ M = T(L)R(\phi)T(d) \]

\[ = \begin{bmatrix}
1 - L\phi & L + d(1 - L\phi) \\
-\phi & 1 - d\phi
\end{bmatrix} \]

\[ q(L) = \frac{L + d(1 - L\phi) + jz_0(1 - L\phi)}{1 - d\phi - j\phi z_0} \]

\[ q(L') = j \frac{z_0}{\phi^2 z_0^2 + (1 - d\phi)^2} = j z_{0-NEW} \]

**ABC**

- Design of ideal imaging systems with geometrical optics
- Gaussian beam propagation

- Rayleigh range?