2

Circuit Theory and Electromagnetics

2.1 Introduction

One of the most important tools of electrical engineers is circuit theory. Circuits have charges and currents, which we know produce electric and magnetic fields. Thus circuits are actually electromagnetic systems and strictly speaking, they should be analyzed starting from the general electromagnetic-field equations, i.e., Maxwell's equations. We start, however, from the two Kirchhoff's laws instead, well aware that circuit theory can be used to accurately predict circuit behavior.

Circuit theory is an approximate theory that can be obtained from Maxwell's equations with a set of approximations. We will return to this point throughout this book. In this chapter we review some simple circuit examples and look at where these approximations are made, arriving at two important conclusions. First, circuit theory is an approximation (but fortunately a very good and useful one in most applications). Second, the limitations of circuit theory can be understood only if we understand electromagnetic-field theory. In the next section we consider the effects of some simple electromagnetic properties of electric circuits, which will help you understand these conclusions. You can perform the examples shown here in the lab, using just a function generator and oscilloscope.

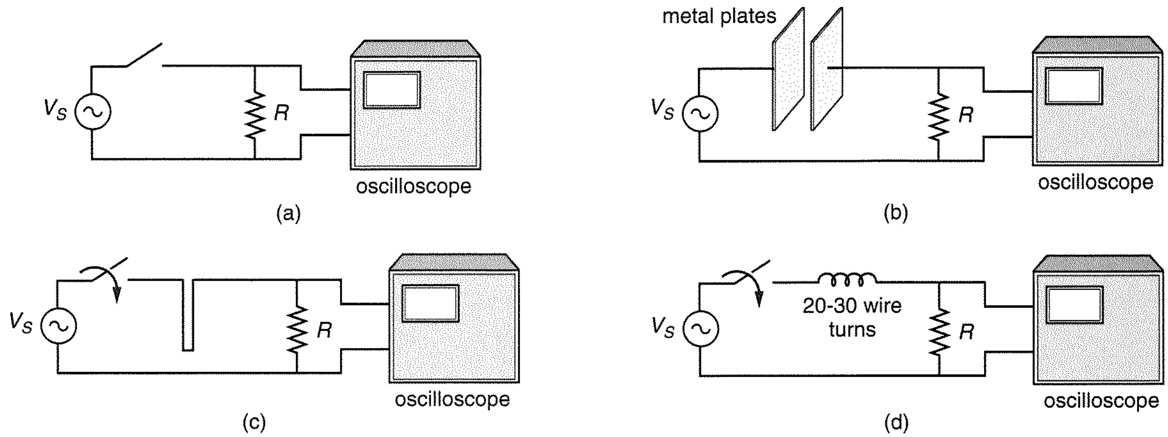

Figure 2.1 (a) A resistor connected to a function generator. The voltage across the resistor is observed on an oscilloscope. (b) The switch in (a) is replaced by two metal plates. (c) An interconnecting conductor is shaped to form a short-circuited two-wire line. (d) The interconnecting conductor is wound around a pen.

2.2 Circuit Elements as Electromagnetic Structures

Let us first consider three basic circuit elements: a switch, a conducting wire, and a resistor (Fig. 2.1). We shall find out later how Ohm's law is derived from Maxwell's equations, but right now let us start from what you learned in circuits: the voltage across the resistor is $v_R(t) = Ri(t)$. (Actually, this relation should be considered as the definition of an *ideal* resistor.)

We connect the resistor to a function generator and look at the voltage across it on an oscilloscope, Fig. 2.1a, when the switch is open. The classical expectation is that the voltage is zero. However, note that the switch consists of two contacts that are separated by an insulator (e.g., air) when the switch is open. These two contacts, therefore, form a capacitor. Only if we can neglect the capacitance of this capacitor, i.e., if we can consider it to be zero, is the voltage across the resistor also zero.

To understand this, imagine changing the *shape* and *size* of the switch. For example, let us replace the actual switch by two parallel rectangular metal plates, say 10 by 10 cm. Let the plates be separated by 1 cm when the switch is open, as in Fig. 2.1b, and pressed tightly together when the switch is closed. The resistance of the large plates is certainly less than that of the switch contacts, and close to 0 Ω, so we should see no change in $v_R(t)$ on the oscilloscope screen. However, in the open position this new switch may influence the current in the circuit considerably because it has a sizable capacitance. Indeed, if we bring the two plates closer together, we will notice on the oscilloscope that this open switch influences the voltage between the resistor terminals more than when the plates are farther apart.

This capacitance is present in any switch, but if—as mentioned—the capacitance is small enough, the switch will *behave* as if the capacitance is zero. However, in strict electromagnetic theory even an element as simple as a switch does not exist. Only if we can neglect its capacitance does a switch behave according to the definition in circuit theory, i.e., that it is either an open switch or a short circuit. To analyze the open switch more accurately, we must consider its capacitance and use electromagnetic-field theory. Recall that the reactance of a capacitor is inversely proportional to the product of its capacitance and frequency. So we may infer that, at extremely high frequencies, it may not be easy to make a switch that, if open, acts indeed as an open circuit.

Let us now concentrate on the influence of the size and shape of a conducting wire. In circuit theory, the wire form and size are assumed to have no effect on circuit behavior and they are assumed to be short-circuit interconnections. We now analyze this assumption in more detail.

Assume that the switch is closed. According to circuit theory, we may vary the length and shape of the wire connecting the resistor to the generator as much as we wish without changing either the voltage across the resistor or current in it. What will happen, however, if we substantially extend one of the conductors and bend it as in Fig. 2.1c, so that we get two relatively long parallel close wires? Experiments show that this changes the voltage across the resistor to a large degree. How can we explain this?

The bent conductor represents a section of short-circuited transmission line. If the frequency is low enough, this is just a wire loop having a certain inductance. This inductance is connected in series with the resistor and changes the current, and hence the voltage across the resistor.

So the circuit-theory assumption that the interconnecting conductors have no effect on the circuit behavior is only an approximation. Electromagnetic-field theory tells us that in the case of varying currents, the same circuit will behave differently if we twist it, extending, shortening, or deforming the interconnecting conductors and generally changing the circuit's shape. This is indeed a strange conclusion if one adheres to circuit-theory explanations, but it is true. At high frequencies, even as low as about 10 MHz, and for circuit dimensions exceeding about 10 cm, circuit theory frequently cannot predict circuit properties with sufficient accuracy, but electromagnetic theory can.

As a more specific example, consider the circuit in Fig. 2.2. It consists of one resistor and one capacitor of very small dimensions (known as "chip" or "surface mount" resistors and capacitors). If we compute the input impedance of the circuit as a function of frequency, we get the solid line in Fig. 2.3. Experimentally obtained results, indicated by the square symbols, are quite different, however. Above a certain

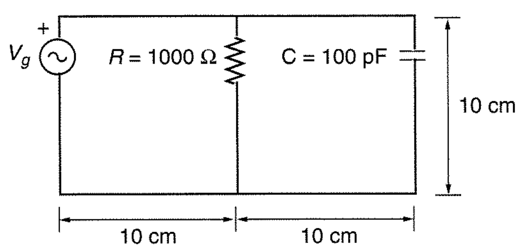

Figure 2.2 A simple circuit with a small resistor and a small capacitor

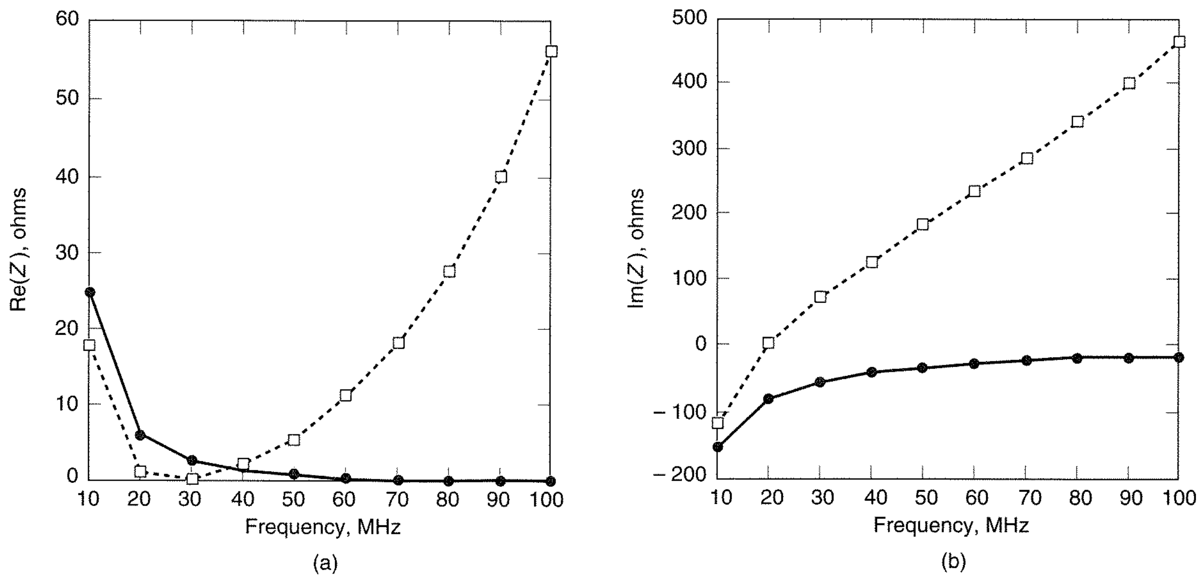

Figure 2.3 Real (a) and imaginary (b) parts of the input impedance of the circuit shown in Fig. 2.2 versus frequency, obtained by circuit theory (solid line with dots), by electromagnetic analysis of the circuit (dashed line), and by experiment (small squares).

frequency, they differ greatly from those predicted by circuit theory. We now know why: in addition to the circuit elements themselves (the resistor and the capacitor), the shape of interconnecting wires in Fig. 2.2 also influences the circuit behavior. This simple circuit can also be analyzed using electromagnetic theory and computer programs that take the shape of the interconnecting conductors into account. The result for the circuit impedance using such a program is shown in Fig. 2.3 in dashed line. You can observe excellent agreement between measurement and theory at frequencies considerably above those where circuit theory loses accuracy. For the moment, you may trust (or not trust) the dashed line results.

Another effect observed in the circuit from Fig. 2.2 is associated with the assumption that the chip (surface mount) components are very small, or *lumped* (which is always assumed in circuit theory). The chip capacitor and resistor in Fig. 2.2 will in reality not have the exact impedance values given in their specification sheets. It turns out that most chip capacitors and resistors have an associated series lead inductance of about 1 nH. That means that above a certain frequency, the chip capacitor will start behaving like an inductor. It is left as an exercise for the reader to calculate this resonant frequency for 1, 10, and 100-pF chip capacitors.

As the next example, let us again close the switch in Fig. 2.1a. Then we take the wire connecting the resistor to the generator and wind it tightly around a pen 20 to 30 times, as shown in Fig. 2.1d. The result is a more conventional inductor than the bent conductor of Fig. 2.1c. Again the shape of the wire has a huge effect on the voltage across the resistor. It has an even larger effect if an iron rod is used instead of the pen. We can find the value of inductance only by using electromagnetic theory, or by measurements. What we will learn in this book is how to take into account the actual

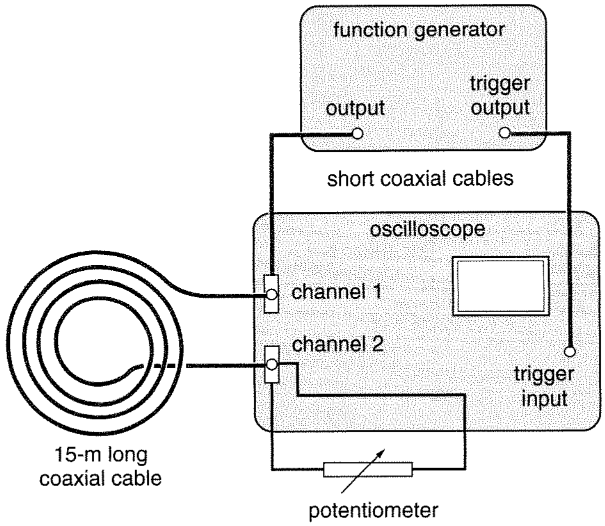

Figure 2.4 Observing electromagnetic effects in a coaxial cable

shape of conductive and nonconductive bodies, and the properties of the materials they are made of, in order to predict the behavior not only of simple circuits but also of different devices used in electrical engineering.

The last example is related to electromagnetic waves and transmission-line theory. Figure 2.4 shows a 15-m coaxial cable connected at one end to a function generator. At the other end it is connected to channel 2 of a two-channel oscilloscope and in parallel with a potentiometer (variable resistor). Channel 1 of the oscilloscope monitors the output of the function generator (which is the input to the coaxial cable). If you used only basic circuit theory, you would expect to see the same voltage for all values of the resistor at the end of the cable, and the voltage should be the same as that coming out of the signal generator. However, due to electromagnetic wave effects, the waveforms at the two channels (the voltages at the beginning and the end of the coaxial cable) can be very different. For example, a 1-V pulse from the signal generator could result in a negative, zero, or greater-than-1-V pulse at channel 2 of the oscilloscope. To explain this result we need so-called transmission-line theory, which turns out to be a special case of electromagnetic wave theory. We shall consider transmission lines in Chapter 18.

Questions and problems Q2.1 to Q2.7, P2.1 to P2.4

2.3 Oscillations in Circuits from the Electromagnetic Point of View

Let us review a more complex (but still very simple) circuit shown in Fig. 2.5, which is a combination of the previous cases. This is a series resonant circuit. For a voltage

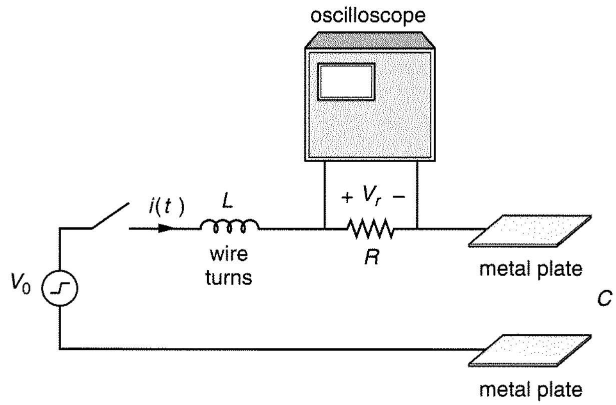

Figure 2.5 A possible physical realization of a series resonant circuit

step V_0 turned on at $t = 0$, Kirchhoff's voltage law gives

$$Ri + L\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{1}{C}\int_0^t i\,\mathrm{d}t + V_0 = 0. \tag{2.1}$$

By differentiating with respect to t and rearranging the terms, we get

$$\frac{\mathrm{d}^2 i}{\mathrm{d}t^2} + \frac{R}{L}\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{i}{LC} = 0, \tag{2.2}$$

which is a second-order ordinary differential equation with exponential solutions of the form

$$i(t) = A_1 \exp(s_1 t) + A_2 \exp(s_2 t). \tag{2.3}$$

A_1 and A_2 are constants determined from the initial conditions, and s_1 and s_2 are complex roots of the characteristic equation of (2.2), given by

$$s_{1,2} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}. \tag{2.4}$$

Let $\omega_0 = 1/\sqrt{LC}$. For $\omega_0^2 > (R/2L)^2$, $s_{1,2} = -\alpha \pm \mathrm{j}\omega$ are complex numbers with real and imaginary parts, as seen in Eq. (2.4), and the solution for the current is

$$i(t) = B_1 e^{-\alpha t}\cos\omega t + B_2 e^{-\alpha t}\sin\omega t. \tag{2.5}$$

The constants $B_{1,2}$ are given by the initial conditions. At $t = 0$ there is no current through the inductor before the voltage is turned on at the input, $i(0) = 0$ and $B_1 = 0$, so $i(t) = B_2 e^{-\alpha t}\sin\omega t$, and B_2 can be found from knowing what $\mathrm{d}i(0)/\mathrm{d}t$ is. Since $i(0) = 0$ immediately after the switch is closed, there is no voltage drop across the resistor and the initial voltage V_0 on the capacitor shows up across the inductor,

$L(\mathrm{d}i/\mathrm{d}t)_{t=0} = V_0$. The final expression for the voltage across the resistor after the switch is closed is

$$v_R(t) = Ri(t) = R\frac{V_0}{L}e^{-\alpha t}\sin\omega t. \tag{2.6}$$

This last expression shows that the voltage is a sinusoid with an exponential amplitude decay. This is called a *damped oscillation*. In electromagnetic terms, the energy in an undamped case is stored in the inductor for one half of the cycle, and in the capacitor in the other half. In a damped case, some of the electromagnetic energy goes into heat in the resistive parts of the circuit.

This effect can also be explained in a similar way by circuit theory. What *cannot* be answered by circuit theory, however, are the following questions:

- Does the resonant frequency depend on the circuit shape and size?
- Does the damping depend on the circuit shape and size?

We already know the answer to the first question: the resonant frequency *does* depend (at least to some extent) on the shape of the circuit. The reasoning is exactly as in the previous examples.

The second question itself seems a bit strange: how can we have more damping than that resulting from losses in the resistor? We mentioned in the first chapter that Maxwell predicted the existence of electromagnetic waves. We will learn that theoretically these waves are produced by *all* systems with time-varying currents, and that the efficiency in producing these waves depends on the system size and shape. We will also learn that an electromagnetic wave is, in fact, an energy package. Thus, "radiation" of electromagnetic waves actually implies leakage, or loss, of energy from the system producing them. Therefore resonant circuits *do* have damping that depends on their size and shape. Fortunately, in most applications this effect is negligible, but it always exists. It can be predicted only by electromagnetic-field theory—circuit theory is unable to do that. We will learn how large a circuit must be to radiate substantially.

Questions and problems Q2.8 and Q2.9, P2.5 to P2.8

2.4 Chapter Summary

1. Circuit theory is not exact; it is an approximation of electromagnetic-field theory.
2. To understand the limitations of circuit theory, we have to begin from electromagnetic-field theory.
3. To determine theoretically the capacitance of a capacitor or the inductance of a coil, it is necessary to use electromagnetic-field theory. The calculation of the resistance of a resistor also requires some knowledge of electromagnetic-field theory.

4. Along transmission lines, such as two-wire or coaxial lines, exist specific electromagnetic waves with specific effects that cannot be explained in terms of circuit-theory concepts.
5. Resonance effects and damping in circuits depend on the circuit shape and size, a strange phenomenon from the circuit-theory viewpoint.

QUESTIONS

Q2.1. Why does every switch have capacitance?

Q2.2. Try to imagine a "perfect" but real switch (a switch with the smallest possible capacitance). How would you design a good switch? What would be its likely limitations?

Q2.3. Why does it become progressively more difficult to have an "ideal" switch as frequency increases?

Q2.4. Why is the circuit-theory assumption that interconnecting conductors (wires) have no effect on the circuit behavior incorrect?

Q2.5. Imagine a resistor connected to a car battery by wires of fixed length. Does the shape of the wires influence the current in the resistor? Explain.

Q2.6. Answer question Q2.5 if the source is a (1) 60-Hz and (2) 1-GHz generator.

Q2.7. Give at least two reasons for the failure of circuit theory when analyzing the simple circuit in Fig. 2.2.

Q2.8. Explain why the resonant frequency of a circuit is at least to some extent dependent on the circuit shape and size.

Q2.9. Why does the damping in resonant circuits depend at least to some extent on the circuits' shape and size? Can circuit theory explain this?

PROBLEMS

P2.1. The capacitance of a switch ranges from a fraction of a picofarad to a few picofarads. Assume that a generator of variable angular frequency ω is connected to a resistor of resistance of $1\,\mathrm{M\Omega}$, but that the switch is open. Assuming a switch capacitance of 1 pF, at what frequency is the open switch reactance equal to the resistor resistance?

P2.2. A surface-mount capacitor has a 1-nH parasitic series lead inductance. Calculate and plot the frequency at which such a capacitor starts looking like an inductor, as a function of the capacitance value.

P2.3. A surface-mount resistor of resistance $R = 100\,\Omega$ has a 1-nH series lead inductance. Plot the real and imaginary part of the impedance as a function of frequency. In which frequency range can this chip be used as a resistor?

P2.4. The windings of a coil have a parasitic capacitance of 0.1 pF, which can be viewed as an equivalent ~~series~~ capacitance. Plot the reactance of such a $1\,\mu\mathrm{H}$ coil as a function of frequency. Parallel

P2.5. A capacitor of capacitance C receives a charge Q. It is then connected to an uncharged capacitor of the same capacitance C by means of conductors with practically no resistance. Find the energy contained in the capacitor before connecting it to the other capacitor, and the energy contained in the two capacitors. [The energy of a capacitor is given by $W_e = Q^2/(2C)$]. Can you explain the results using circuit-theory arguments? Can you explain the results at all?

P2.6. The inductance of a thin circular loop of radius R, made of wire of radius a, where $R \gg a$, is given by the approximate formula

$$L_0 \simeq \mu_0 R \left(\ln \frac{8R}{a} - 2 \right) \text{ (henries)},$$

where $\mu_0 = 4\pi\, 10^{-7}$ H/m, and R and a are in meters. A capacitor of capacitance $C = 100\,\text{pF}$ and a coil of inductance $L = 100\,\text{nH}$ are connected in series by wires of radius a and the shape of a circular loop of radius R ($R \gg a$). Find: (1) the radius of the loop that results in $L_0 = L$ if $a = 0.1\,\text{mm}$; (2) the radius of the wire that results in $L_0 = L$ if $R = 2.5\,\text{cm}$; and (3) the resonant frequency of the circuit versus the loop radius, R, if $a = 0.1\,\text{mm}$.

P2.7. The capacitance between the terminals of a resistor is $C = 0.5\,\text{pF}$, and its resistance is $R = 10^6\,\Omega$. Plot the real and imaginary part of the impedance of this dominantly resistive element versus frequency from 0 MHz to 10 MHz.

P2.8. A coil is made in the form of $N = 10$ tightly packed turns of wire. Predict qualitatively the high-frequency behavior of the coil. Explain your reasoning.